# Contents

# Introduction

The following work will investigate discretizations (see also summary) of the socalled (two dimensional) sine-Gordon equation :

$$\omega_{tt} - \omega_{xx} = ksin\omega$$

or it's reduction to the socalled pendulum equation:

$$\omega_{tt} = ksin\omega$$

The sine-Gordon equation has many important applications in physics. One of the most well known examples is that of a pendulum chain, where the sine-Gordon equation describes a chain of coupled pendula rotating in a plane [24]. The sine-Gordon equation serves also as one of the simplest nonlinear field theories, where it is of special interest that in the weak field limit $|\omega| \ll 1$ one recovers the Klein-Gordon equation

An important feature of the sine-Gordon equation is that it admits solutions which are - in contrary to plane wave solutions - localized in space time, socalled solitons, or solitary waves. Nonlinear equations which show this feature are therefore called soliton equations. The study of soliton equations has produced a whole new branch in mathematics and physics. For an excellent survey by two people, who contributed also a very big part to this area, see the work of Faddeev and Takhdajan [19].

But also in geometry, the sine-Gordon equation has its application. It can be viewed as the integrability condition for surfaces with constant negative Gaussian curvature, socalled K-surfaces, which were already studied by Bianchi [40]. The normal map of such surfaces appears then again in physics as the socalled $O^3$ invariant chiral model [44].

After having described the importance of the sine-Gordon equation in mathematics and physics the next question to ask is why study discretizations. Here we enter a more modern domain.

As already indicated Soliton equations describe nonlinear phenomena. In order to reconstruct these phenomena as there are e.g. flood waves, shallow water waves, turbulences in fluid mechanics, etc. they need to be modeled on a computer. But by construction a computer always needs a discrete version of a differential equation. Therefore it is of certain importance to know how good a discretization is and which features appearing in the continous theory can be recovered in it's corresponding discrete theory.

Another reason for discretizing, espiacially in the sine-Gordon case is that when one quantizes a classical equation, i.e. if one tries to substitute the classical variables (fields) by operators on some Hilbertspace then one usually gets the problem of divergencies. This can't happen in a discrete theory.

A good way to control the discretizations of a classical equation is to take a detour over differential geometry; i.e. to buid a discrete surface. In the case of the sine-Gordon theory this means to rebuild K-surfaces - a task which was started by the Vienna school with W. Wunderlich [57] and which was accomplished by the Berlin school with A. Bobenko and U. Pinkall [7],[5] who drew in particular a connection to soliton theory.

**The present work is organized as follows.**

The **first chapter** and **second chapter** will be concerned with the classical theory.

The **first chapter** reviews the theory of K-surfaces and of discrete K-surfaces and their relation to the Volterra model and the $O^3$ invariant chiral model.

The **second chapter** investigates the classical dynamics of various discretizations of the sine-Gordon equation. An emphasis will be put on the investigation of the lagrangian structure of the corresponding difference equations. In particular a Lagrangian action for sine-Gordon variables assigned to the vertices of the space-time lattice will be introduced. Using covariant phase space techniques one can derive from this lagrangian action an unique nondegenerate translational invariant poisson structure. The reduction of the vertex variables to their difference (with constant monodromy), which gives variables assigned to the faces of the space-time lattice, can be interpreted as a phase space reduction (Marsden-Weinstein reduction). Moreover it will be shown that the vertex variables themselves can be obtained from a bigger algebra namely an algebra which is generated by variables which are assigned to the edges of the space-time lattice. These "edge variables" describe (modulo redefinition along the diagonals) the discrete changes of a frame when moving along a discrete K-surface, i.e. the discrete version of $L = \phi d\phi$. It will be shown that the reduction from the edge variables to the vertex variables refers to a gauge fixing of the frame $\phi$. It will also be shown that this is gauge fixing is related to a choice of poisson relations on the edge algebra, a fact which will become important in the next chapter.

The **third chapter** is dedicated to the study of the quantized models of the first chapter.

Its **first part** gives a short introduction to what is meant by the term "quantization".

In the **second part** quantum evolution equations on all the in chapter one involved algebras will be derived. In particular it will turn out that a subalgebra of the above mentioned edge algebra forms a fermionic algebra, i.e. it's generators satisfy canonical anticommutation relations (CAR). The quantum evolution on that subalgebra will be that of free massive fermions. For a full description of the evolution on this subalgebra the introduction of socalled lightcone shifts, which can be envisaged as quantized symplectomorphisms on phase space will be necessary. As it turns out the introduction of these shifts results in a fixing of the roots of an equation, which is usually called the classical background equation [52]. [39]

The **third part** of chapter three deals with the construction of integrals of motion for the quantum sine-Gordon model. In particular the explicit form of these integrals for the reduced case of a quantum pendulum will be derived. The only nontrivial integral to obtain in this case will be called hamiltonian of the quantum pendulum. It depends on two parameters $\alpha, k \in \mathbb{R}$. We will study this hamiltonian in finite dimensional representations. In the case, where $e^{i\alpha} = q$, where q more or less labels the different finite dimensional representations the hamiltonian of the quantum pendulum will be up to a constant the square of the socalled Hofstadter hamiltonian. Both hamiltonians have important applications in solid state physics (see e.g. [31, 33, 55, 17, 18, 3, 37, 35])). The analytic determination of the spectrum of the Hofstadter hamiltonian, as well as quantum pendulum is an unsolved problem and hence one has to describe the spectrum by other means.

After a short description of the method of Bethe ansatz it will be shown that one can express the spectrum of the quantum pendulum for all $e^{i\alpha} = q^l$, $l \in \mathbb{Z}$ by a set of Bethe ansatz equations.

In the **fourth chapter** some more applications of the discrete sine-Gordon and pendulum equation will be given.

It will be shown that the equations which govern the evolution of a particle in a billard with an ellipse as boundary will be more or less identical to the ones for the discrete pendulum.

It will be shown that the heights of the normals of a rotational symmetric K-surface satisfy the square root of the pendulum equation. Since in the quantization chapter also a quantization for the square root of the pendulum equation was suggested, one may hence investigate how with this information gives a possible quantization for a rotational symmetric $O^3$-invariant chiral model. The pendulum equation for the squares of the heights of the normals belonging to a K-surface is unfortunately a reduction of a sine-Gordon type equation, which in general can't be related to a K-surface but belongs to another geometrical model. A description of this model will be given.

# Chapter 1

# Geometry of K-surfaces

A **K-surface** is a $n$ dimensional smooth surface with constant Gaussian curvature $K = -1$.

In the following we will only consider two dimensional K-surfaces (From now on briefly K-surfaces). All (two dimensional) surfaces with Gaussian curvature $K = -1$ can be described in certain parametrizations, which posess an obvious discrete analog. Hence we will extend the definition of a (two dimensional) smooth K-surface to the before mentioned discrete analog, which will be called (two dimensional) **discrete K-surface** or if the context is clear also just K-surface.

We will give now a brief review of smooth (two dimensional) K-surfaces and introduce the above mentioned parametrizations in order to motivate the definition of discrete K-surfaces.

Readers which are familiar with the definition of K-surfaces can skip sections 1.1 and 1.2.

The connection between discrete K-surface and the socalled Volterra model will be given after that section.

## 1.1 Smooth K-surfaces

### 1.1.1 Smooth K-surfaces in terms of classical geometry

Let $\mathcal{M}$ be a two dimensional smooth orientable manifold immersed into $\mathbb{R}^3$. Let $\tilde{F} : U \subset \mathcal{M} \to \mathbb{R}^3$ be a parametrization of $\mathcal{M}$. The metric $g^{\mathcal{M}}$ on $\mathcal{M}$ shall be induced, i.e. $g^{\mathcal{M}} = \langle d\tilde{F}, d\tilde{F} \rangle^{\mathbb{R}^3}$. Then there exists a smooth map, called Gaussmap $\tilde{N} : \mathcal{M} \to S^2 \subset \mathbb{R}^3$ which sends each point of the manifold to the tip of the unit normal along $\tilde{F}(\mathcal{M})$. Often the Gaussmap is defined as the map $N := \tilde{N} \circ \vec{x}^{-1} : \vec{x}(U) \subset \mathbb{R}^2 \to S^2$ where $(U, \vec{x})$ is a local chart of $\mathcal{M}$. We will mostly work with this definition of a Gaussmap. Define analogously $F = \tilde{F} \circ \vec{x}^{-1} : \vec{x}(U) \subset \mathbb{R}^2 \to \mathbb{R}^3$.

The differential $d\tilde{N}_p : \mathcal{T}_p\mathcal{M} \to \mathcal{T}_p S^2 \approx \mathcal{T}_p\mathcal{M} \subset \mathbb{R}^3$ of the Gaussmap defines a symmetric bilinear form $II : \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$ by:

$$II_p(X,X) = -\langle d\tilde{F}(X), d\tilde{N}(X)\rangle^{\mathbb{R}^3}_{|p} \qquad X \in \mathcal{T}_p\mathcal{M};$$

$II : \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$ is called **second fundamental form**.

The determinant of the linear operator $d\tilde{F}^{-1} \circ d\tilde{N}_p : \mathcal{T}_p\mathcal{M} \to \mathcal{T}_p\mathcal{M}$ is called Gaussian curvature $K$ of $\mathcal{M}$ at $p$. If we express the second fundamental form with respect to a local basis, i.e $II = edx^2 + fdxdy + gdy^2$ and analogously for the induced metric $I$ (which is also called first fundamental form),i.e. $I = Edx^2 + Fdxdy + Gdy^2$, then :

$$K = \frac{detII}{detI} = \frac{eg - f^2}{EG - F^2} \tag{1.1.1}$$

If $K : \mathcal{M} \to \mathbb{R}$ is negative and nonvanishing then it is straightforward to show that there exist two linear independent vector fields $U, V \in \mathcal{X}(\mathcal{M})$ such that

$$II_p(U) = II_p(V) = 0 \qquad \text{f. a.} \quad p \in \mathcal{M}.$$

The integral curves of the vector fields $U$ and $V$ are called **asymptotic lines**. An **asymptotic line parametrization** is henceforth a parametrization $F : (x,y) \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$II_p(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = II_p(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = -\langle F_x, d\tilde{N}(\frac{\partial}{\partial x})\rangle_{\mathbb{R}^3} = -\langle F_y, d\tilde{N}(\frac{\partial}{\partial y})\rangle_{\mathbb{R}^3} = 0$$

It follows that

$$0 = \langle F_x, d\tilde{N}(\frac{\partial}{\partial x})\rangle = \langle F_x, N_x\rangle = \frac{\partial}{\partial x}\langle F_x, N\rangle - \langle F_{xx}, N\rangle = -\langle F_{xx}, N\rangle,$$

where $F_x = \frac{\partial}{\partial x}F$ a.s.o. Analogously $\langle F_{yy}, N\rangle = 0$

Hence besides the tangential vectors $F_x, F_y$ (at a point p) also the vectors $F_{xx}, F_{yy}$ are orthogonal to $N$, i.e. in particular the planes spanned by $F_x$ and $F_{xx}$, which are called **osculating planes** along the coordinate line $F(x, y = const)$ are parallel to the tangent planes of $\mathcal{M}$ along $F(x, y = const)$. The same holds for the line $F(x = const, y)$. The second fundamental form has in this case only more off diagonal entries, namely:

$$II = -\langle d\tilde{F}, d\tilde{N}\rangle = 2\langle F_{xy}, N\rangle dxdy$$

Let $F : (x,y) \subset \mathbb{R}^2 \to \mathbb{R}^3$ be from now on an asymptotic line parametrization. Define

$$|F_x| = A, \qquad |F_y| = B$$

and denote with $\omega$ the angle between the asymptotic lines on the surface, i.e.:

$$\langle F_x, F_y\rangle_{\mathbb{R}^3} = AB\cos\omega$$

In addition let us demand that the Gaussian curvature is $K = -1$. Using (1.1.1) we get for the first fundamental form $I$ (the metric) and the second fundamental form $II$ of a surface in asymptotic line parametrization with $K = -1$:

$$\begin{aligned} I &= \langle d\tilde{F}, d\tilde{F}\rangle^{\mathbb{R}^3} = A^2dx^2 + 2AB\cos\omega dxdy + B^2dy^2 \\ II &= -\langle d\tilde{F}, d\tilde{N}\rangle^{\mathbb{R}^3} = 2AB\sin\omega dxdy. \end{aligned}$$

The zero curvature of euclidean space $\mathbb{R}^3$:

$$D_XD_Yd\tilde{F}(Z) - D_YD_Xd\tilde{F}(Z) - D_{[X,Y]}d\tilde{F}(Z) = 0$$

induces via

$$D_Xd\tilde{F}(Y) = d\tilde{F}(\nabla^{\mathcal{M}}_X Y) + II(X,Y)\tilde{N}$$

an identity, which involves only the first and second fundamental forms of $\mathcal{M}$. The tangential and normal components of this identity give the Gauss and Mainardi-Codazzi equations (shortly called Gauss-Codazzi equations). The Gauss-Codazzi equations serve as integrability conditions (see also Theorem 1.1.4 below).

In the above described special case of a surface in asymptotic line parametrization with the additional demand that $K = -1$ the Gauss-Codazzi equations are:

$$\omega_{xy} - AB\sin\omega = 0 \qquad (1.1.2)$$

$$A_y = B_x = 0 \qquad (1.1.3)$$

**Definition 1.1.1** *A parametrization* $F : (x,y) \subset \mathbb{R}^2 \to \mathbb{R}^3$ *satisfying*

$$A_y = B_x = 0$$

*is called a* **weak Chebychev parametrization** *(or weak Chebychev net); if in addition*

$$|F_x| = A = \mathit{const} \in \mathbb{R}\backslash\{0\} \quad |F_y| = B = \mathit{const} \in \mathbb{R}\backslash\{0\}$$

*(constant length tangent vectors)*

a. $A \neq B$ *the above parametrization is called an* **anisotropic Chebychev parametrization***(net)*

b. $A = B$ *the above parametrization is called a* **Chebychev parametrization***(net).*

One can show that any surface which is simultanously parametrized in a (weak) Chebychev and asymptotic line parametrization is a surface with constant Gaussian curvature $K = -1$ (K-surface). It follows that the angle between the asymptotic lines satisfies equation (1.1.2) which is called **sine-Gordon equation**.

In addendum we remark the following:

**Theorem 1.1.2** *(see e.g. [7]) Every surface with constant negative Gaussian curvature posesses a one-parameter family of deformations preserving the second fundamental form, the Gaussian curvature and the angle $\omega$ between the asymptotic lines. The deformation is given by*

$$A \to \lambda A \qquad B \to \frac{1}{\lambda} B$$

$\lambda$ *will be called* **spectral parameter**.

### 1.1.2 The Gauss map of a smooth K-surface

**Proposition 1.1.3** *([7] The Gauss map $N : \mathbb{R}^2 \to S^2$ of a smooth surface with Gaussian curvature $K = -1$ is lorentzharmonic,i.e:*

$$N_{xy} = \rho N \qquad \rho : \mathbb{R}^2 \to \mathbb{R} \tag{1.1.4}$$

From (1.1.4) it follows immediately that

$$|N_x| =: A, \qquad |N_y| =: B.$$

We call such a parametrization of the Gauss map $N : \mathbb{R}^2 \to S^2$ of a smooth K-surface forms a lorentzharmonic (weak) Chebychev net on $S^2$. Moreover let

$$\omega := arccos(\frac{1}{AB}\langle N_x, N_y\rangle_{\mathbb{R}^3}). \tag{1.1.5}$$

We obtain

$$\begin{aligned} N_{xx} &= -A^2 N + \omega_x \frac{\cos\omega}{\sin\omega} N_x - \frac{A}{B}\omega_x \frac{1}{\sin\omega} N_y \\ N_{yy} &= -B^2 N + \omega_y \frac{\cos\omega}{\sin\omega} N_y - \frac{B}{A}\omega_y \frac{1}{\sin\omega} N_x \end{aligned}$$

Inserting this into (1.1.4) gives us the sine-Gordon equation in $\tilde{\omega} = \pi - \omega$:

$$\omega_{xy} + AB\sin\omega = 0$$

the Gauss map $N : \mathbb{R}^2 \to S^2$ of a smooth K-surface is also wellknown in physics as the $O_3$ invariant Chiral model [44].

### 1.1.3 Smooth K-surfaces in $su(2)$ description

We identify $R^3$ with the space of imaginary quaternions

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \leftrightarrow x = -i\sum_{k=1}^{3} x_k \sigma_k \in su(2)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The scalar product in $R^3$ reads as $\langle x, y\rangle_{\mathbb{R}^3} = -\frac{1}{2}trXY$. Let $D(A, e^{i\phi}) := Re(e^{i\phi/2})\mathbb{1} + Im(e^{i\phi/2}A) \in SU(2)$. A rotation $\mathcal{R}(a, e^{i\phi}) \in SO(3)$ of a vector $x$ around a unit vector $a$ with angle $\phi$ is then given by:

$$\mathcal{R}(a, e^{i\phi})x = D(A, e^{i\phi})^{-1} X D(A, e^{i\phi})$$

**Theorem 1.1.4** *([7]) Let $F : (u, v) \subset F : \mathbb{R}^2 \to \mathbb{R}^3 = su(2)$ be a surface of constant negative Gaussian curvature $K = -1$ and $\Phi(u, v) \in SU(2)$ a unit quaternion which transforms the basis $\{E_i\}_{i=1,2,3}$ in $\mathbb{R}^3 = su(2)$:*

$$\begin{aligned} E_1 &= -i\lambda A(\cos\frac{\phi}{2}\sigma_1 + \sin\frac{\phi}{2}\sigma_2) \\ E_2 &= -i\frac{1}{\lambda}B(\cos\frac{\phi}{2}\sigma_1 - \sin\frac{\phi}{2}\sigma_2) \\ E_3 &= -i\sigma_3 \end{aligned}$$

*into the basis*

$$\begin{aligned} F_u &= \Phi^{-1}E_1\Phi \\ F_v &= \Phi^{-1}E_2\Phi \\ N &= \Phi^{-1}E_3\Phi. \end{aligned}$$

*Then $\Phi$ satisfies the system*

$$\Phi_u = U\Phi, \qquad \Phi_v = V\Phi \tag{1.1.6}$$

*where*

$$\begin{aligned} U &= \frac{i}{2}\begin{pmatrix} \frac{\phi_u}{2} & -A\lambda e^{-i\phi/2} \\ -A\lambda e^{i\phi/2} & -\frac{\phi_u}{2} \end{pmatrix} \\ V &= \frac{i}{2}\begin{pmatrix} -\frac{\phi_v}{2} & \frac{B}{\lambda}e^{i\phi/2} \\ \frac{B}{\lambda}e^{-i\phi/2} & \frac{\phi_v}{2} \end{pmatrix}. \end{aligned} \tag{1.1.7}$$

*The so called zero-curvature condition:*

$$U_v - V_u + [U, V] = 0 \tag{1.1.8}$$

*gives the Gauss-Codazzi equations (1.1.2),(1.1.3) of the surface.*
*On the other hand if $\Phi(u, v) \in SU(2)$ is a solution to (1.1.6) then*

$$F = 2\Phi^{-1}\frac{\partial}{\partial t}\Phi \qquad \lambda = e^t \qquad \textit{Sym's Formula}$$

*is a surface of constant negative curvature $K = -1$ with the fundamental forms*

$$\begin{aligned} I &= \langle d\tilde{F}, d\tilde{F}\rangle = A^2 + 2AB\cos\omega dxdy + B^2dy^2 \\ II &= -\langle d\tilde{F}, d\tilde{N}\rangle = 2AB\sin\omega dxdy. \end{aligned}$$

*Its Gauss map is given by $N = -i\Phi^{-1}\sigma_3\Phi$.*

## 1.2 Discrete K-surfaces

**Definition 1.2.1** *A* **discrete surface** *is a map* $F : (n, m) \subset \mathbb{Z}^2 \to \mathbb{R}^3$ *with*

$$F : (n, m) \mapsto F(n, m) = F_{n,m}$$

*An* **edge** *on a surface is the vector*

$$F_{n_1,m_1}^{n_2,m_2} := F_{n_2,m_2} - F_{n_1,m_1} \in \mathbb{R}^3$$

*where* $|n_1 - n_2| = 1$ *exclusive or* $|m_1 - m_2| = 1$*. Sometimes* $F_{n_1,m_1}^{n_2,m_2}$ *will be used for denoting the ordered tupel* $(F_{n_2,m_2}, F_{n_1,m_1})$*.*
*A* **face** *on a surface is the equivalence class of all ordered quadrupels*

$$(F_{n,m}, F_{n,m+1}, F_{n+1,m+1}, F_{n+1,m})$$

*modulo the equivalence relation of cyclic permutation.*
*If the points* $F_{n-1,m}, F_{n,m+1}, F_{n+1,m}, F_{n,m}, F_{n,m-1}$ *lie all in one plane* $\mathcal{P}_{n,m}$ *(as points in* $\mathbb{R}^3$*) then the normal vector at a point* $F_{n,m}$ *is given by*

$$N_{n,m} = F_{n,m}^{n+1,m} \times F_{n,m}^{n,m+1} \tag{1.2.9}$$

*The map* $N : (n, m) \subset \mathbb{Z}^2 \to \mathbb{R}^3$ *will be called a* **discrete Gaussmap**.

**Definition 1.2.2** *A* **discrete K-surface** *is a discrete surface*

$$F : (n, m) \subset \mathbb{Z}^2 \to \mathbb{R}^3$$

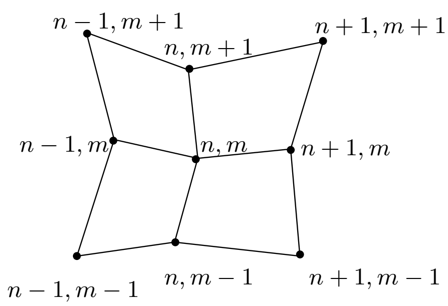
*for which*

a. *For each point* $F_{n,m}$ *there is a plane* $\mathcal{P}_{n,m}$ *such that*

$$F_{n-1,m}, F_{n,m+1}, F_{n+1,m}, F_{n,m}, F_{n,m-1} \in \mathcal{P}_{n,m}$$

b. *The lengths of opposite edges of a face are equal, i.e. the surface is parametrized in a (weak) discrete Chebychev net in* $\mathbb{R}^3$*:*

$$\begin{aligned} |F_{n+1,m}^{n,m}| = |F_{n+1,m+1}^{n,m+1}| &=: A_n \neq 0 \\ |F_{n,m+1}^{n,m}| = |F_{n+1,m+1}^{n+1,m}| &=: B_n \neq 0 \end{aligned}$$



Hence the definition of a K-surface implies the equality of opposite angles of a face.

In section 1.1.2 we learned that the Gauss map of a smooth K-surface forms a lorentzharmonic (possibly weak) Chebychev net on $S^2$. The following proposition shows that the same holds for the discrete case, where the definitions of a discrete (weak) Chebychev net on $S^2$ and the notion of discrete Lorentzharmonicity will be given first.

**Definition 1.2.3** *A* **discrete (weak) Chebychev net on** $S^2$ *is a map*

$$\begin{array}{rcl} N:\mathbb{Z} & \longrightarrow & S^2 \subset \mathbb{R}^3 \\ (n,m) & \mapsto & N_{n,m} \end{array}$$

*such that*

$$\begin{array}{rcccl} \langle N_{n,m}, N_{n+1,m}\rangle & = & \langle N_{n,m+1}, N_{n+1,m+1}\rangle & = & \cos\delta_{1m} \\ \langle N_{n,m}, N_{n,m+1}\rangle & = & \langle N_{n+1,m}, N_{n+1,m+1}\rangle & = & \cos\delta_{2n} \end{array} \tag{1.2.10}$$

*i.e. the length of opposite arcs of a spherical quadrilateral are equal.*

**Definition 1.2.4** *A map* $N:\mathbb{Z}^2 \to \mathbb{R}^3$ *for which*

$$N_{n+1,m+1} - N_{n+1,m} - N_{n,m+1} + N_{n,m} = \rho_{nm}(N_{n+1,m+1} + N_{n+1,m} + N_{n,m+1} + N_{n,m}) \tag{1.2.11}$$

*with* $\rho_{n,m}:\mathbb{Z}^2 \to \mathbb{R}$ *is called* **lorentzharmonic**.

**Lemma 1.2.5** *(see [7]) A lorentzharmonic discrete (weak) Chebychev net on* $S^2$ *is given by its values along an initial Cauchy zig-zag on* $S^2$*, i.e. a sequence of initial conditions of the form* $(\ldots N_{n,m+1}, N_{n,m}, N_{n+1,m}, N_{n+1,m-1}\ldots)$ *and the discrete equation*

$$N_{n+1,m+1} = 2\frac{\langle N_{n+1,m} + N_{n,m+1}, N_{n,m}\rangle}{\|N_{n+1,m} + N_{n,m+1}\|^2}(N_{n+1,m} + N_{n,m+1}) - N_{n,m} \tag{1.2.12}$$

**Proposition 1.2.6** *(see [7]) The Gauss map of a discrete K-surface is a lorentzharmonic discrete (weak) Chebychev net on* $S^2$*. Any lorentzharmonic weak Chebychev net in* $S^2$ $N:\mathbb{Z}^2 \to S^2 \subset \mathbb{R}^3$ *is the Gauss map of a discrete K-surface, which is determined by* $N$ *uniquely up to homothety and translations.*

*The edges of the discrete K-surface are given by*

$$\begin{array}{rcl} F_{n+1,m} - F_{n,m} & = & const \cdot N_{n+1,m} \times N_{n,m} \\ F_{n,m+1} - F_{n,m} & = & const \cdot N_{n,m} \times N_{n,m+1}, \qquad const \in \mathbb{R}. \end{array}$$

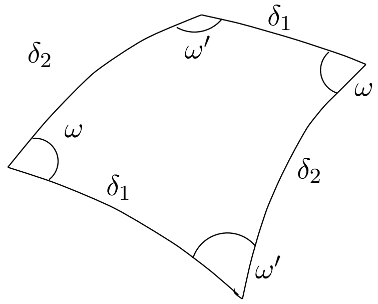
From the above propostion 1.2.6 it follows immediately that the angle $\omega$ between two arcs of a spherical quadilateral is $\pi - \tilde{\omega}$, where $\tilde{\omega}$ is the angle between two edges of a face on th K-surface:

$$\begin{aligned}
\omega &= \angle(arc(N_{n,m+1}, N_{n,m}), arc(N_{n,m}, N_{n+1,m}) \\
&= \angle(N_{n,m} \times N_{n+1,m}, N_{n,m} \times N_{n,m+1}) \\
&= \angle(F_{n+1,m} - F_{n,m}, -(F_{n,m+1} - F_{n,m})) \\
&= \pi - \angle(F_{n+1,m} - F_{n,m}, (F_{n,m+1} - F_{n,m})) \\
&= \pi - \tilde{\omega}.
\end{aligned}$$

Comparing with section 1.1.2 we find that this holds also in the smooth case.

### 1.2.1 Spherical geometry, Derivation of the sine-Gordon equation

By definition 1.2.10 we know that the length of two opposite sides of a spherical quadrilateral or face have to be the same. Hence also opposite angles have to be the same.




By Napiers rule ([8] p. 207) we know that

$$\cot\frac{\omega}{2}\cos\frac{\delta_1-\delta_2}{2} = \tan\frac{\omega'}{2}\cos\frac{\delta_1+\delta_2}{2}$$

$\Leftrightarrow$

$$\tan\frac{\omega}{2}\tan\frac{\omega'}{2} = \frac{\cos\frac{\delta_1}{2}\cos\frac{\delta_2}{2}+\sin\frac{\delta_1}{2}\sin\frac{\delta_2}{2}}{+\cos\frac{\delta_1}{2}\cos\frac{\delta_2}{2}-\sin\frac{\delta_1}{2}\sin\frac{\delta_2}{2}}$$

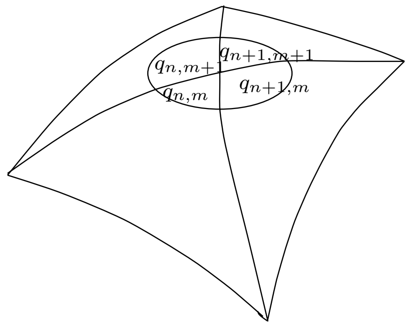
Defining $k = \tan\frac{\delta_1}{2}\tan\frac{\delta_2}{2}$ we obtain

$$\tan\frac{\omega}{2}\tan\frac{\omega'}{2} = \frac{1+k}{1-k}$$

Inserting the definition $q = e^{i\omega}$ we finally obtain

$$q' = -\frac{1+kq}{k+q}$$

Considering four neighbouring quadrilaterals the angles at the centerpoint $P$ have to sum up to $2\pi$, i.e. $q_{n+1,m+1}q_{n,m}q'_{n+1,m}q'_{m,n+1} = 1$.




Hence we obtain an evolution for the angles between two spherical edges:

$$q_{n+1,m+1}q_{n,m} = (q'_{n+1,m})^{-1}(q'_{m,n+1})^{-1} = \frac{q_{n+1,m} + k_{n+1,m}}{1 + k_{n+1,m}q_{n+1,m}} \frac{q_{m,n+1} + k_{m,n+1}}{1 + k_{m,n+1}q_{m,n+1}} \tag{1.2.13}$$

If we express the angles between the edges of a spherical quadrilateral by the angles between the edges of a discrete K-surface (1.2.13),i.e. $\tilde{\omega} = \pi - \omega$ then we get from (1.2.13):

**Proposition 1.2.7** *([7]) The angles between the edges of a discrete K-surface, which correspond in the smooth case to the angles between the asymptotic lines of a surface satisfy the following equation:*

$$\tilde{q}_{n+1,m+1}\tilde{q}_{n,m} = \frac{\tilde{q}_{n+1,m} - k_{n+1,m}}{k_{n+1,m}\tilde{q}_{n+1,m} - 1} \frac{\tilde{q}_{m,n+1} - k_{m,n+1}}{k_{m,n+1}\tilde{q}_{m,n+1} - 1} \tag{1.2.14}$$

*where*

$$k_{n,m}k_{n+1,m+1} = k_{n+1,m}k_{n,m+1}$$

Hence (1.2.13) is a discrete version of the sine-Gordon equation (1.1.2). Althaugh (1.2.14) differs by a sign from (1.2.13) we will sometimes refer also to (1.2.14) as being "the" discrete sine-Gordon equation.

### 1.2.2 Discrete surfaces in su(2) description

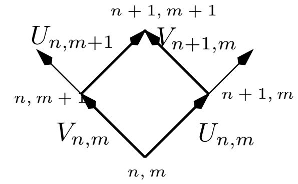
As in the smooth case (1.1.4) we define a the matrix $\phi_{n,m}$ which rotates a normal vector positioned in $z$-direction to the position it should assume at the point $F_{n,m}$ of our surface:

$$N_{n,m} = -i\phi_{n,m}^{-1}\sigma_3\phi_{n,m} \qquad \phi_{k,t} \in SU(2) \tag{1.2.15}$$

similar as in (1.1.6) $\phi_{n,m}$ shall be the at the point $(n, m) \in \mathbb{Z}^2$ evaluated solution $\phi : \mathbb{Z}^2 \to SU(2)$ of the initial value problem

$$\phi_{n+1,m} := U_{n,m}\phi_{n,m} \tag{1.2.16}$$

$$\phi_{n,m+1} := V_{n,m}\phi_{n,m} \qquad U_{n,m}, V_{n,m} \in Su(2) \quad \phi_{0,0} \in SU(2) \tag{1.2.17}$$




where in analogy to the zero curvature in the smooth case (1.1.8) we impose the compatibility condition:

$$U_{n,m}V_{n,m}^{-1} = V_{n+1,m}^{-1}U_{n,m+1} \tag{1.2.18}$$

which ensures that definitions 1.2.17 are welldefined. For the discussion in the following it will be useful to introduce another notation. We numerate our lattice in $\mathbb{Z}^2$ as depictured in the figure 1.

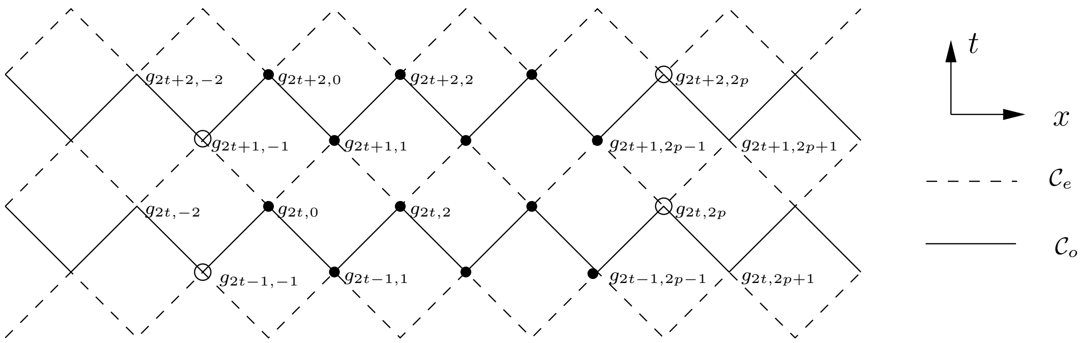



Figure 1

The matrix solutions $\phi$ will be indexed accordingly. The indices $x$ and $t$ will be usually called time and position indices for reasons, which will become apparent later. In the same way we will usually call the above lattice a **Minkowski space time** .

The newly introduced notation has the advantage that we can distinguish the transport matrices $U_{n,m}, V_{n,m}$ solely by their footpoints on the lattice. Define:

$$\begin{aligned} T_{t,k} &:= \phi_{t,k}\phi_{t-1,k-1}^{-1} \quad \text{if} \quad k-t \quad \text{even} \\ T_{t,k} &:= \phi_{t-1,k}\phi_{t,k-1}^{-1} \quad \text{if} \quad k-t \quad \text{odd} \end{aligned}$$

(Note that $\phi_{i,j}$ is only defined if $i-j$ even). So we find:

$$\begin{aligned} T_{t,k} &\simeq U_{t,k} \quad \text{for} \quad k-t \quad \text{even} \\ T_{t,k} &\simeq V_{t,k}^{-1} \quad \text{for} \quad k-t \quad \text{odd} \end{aligned}$$

The conditions

$$\begin{aligned} \cos\delta^1_{k-t} &= \langle N_{t+1,k}, N_{t,k+1}\rangle \quad \text{if} \quad k-t \quad \text{odd} \\ \cos\delta^2_{k+t+2} &= \langle N_{t,k}, N_{t+1,k+1}\rangle \quad \text{if} \quad k-t \quad \text{even} \end{aligned}$$

where $N_{t,k} = -i\phi_{t,k}^{-1}\sigma_3\phi_{t,k}$ together with the fact that $T_{t,k} \in Su(2)$ allow us to parametrize

$$T_{t,k} = \cos\frac{\delta_j^i}{2}\begin{pmatrix} e^{i\alpha_{t,k}} & ie^{i\omega_j^i}e^{i\beta_{t,k}} \\ ie^{i\omega_j^i}e^{-i\beta_{t,k}} & e^{-i\alpha_{t,k}} \end{pmatrix}$$

where

$$e^{\omega_j^i} = \tan\frac{\delta_j^i}{2} \quad \text{and} \quad \left\{ \begin{array}{llll} i=1 & j=k-t & \text{if} & k-t \quad \text{odd} \\ i=2 & j=k+t & \text{if} & k-t \quad \text{even} \end{array} \right\}$$

We know from (1.1.2) that the Gauss equation should be invariant with respect to the transformation $\omega_j^i \to \omega_j^i(\lambda)$, where:

$$\omega_j^i(\lambda) := (-1)^i\lambda + \omega_j^i \tag{1.2.19}$$

$\lambda$ is called a **spectral parameter**. The following proposition shows that the introduction of the spectral parameter $\lambda$ ensures the lorentzharmonicity of the corresponding normal vectors.

**Proposition 1.2.8** *(see also [7]) If we do not specify any boundary conditions then any choice of the matrices $T_{t,k}$ of the form*

$$T_{t,k}(\lambda) = \cos\frac{\delta_j^i}{2}(\lambda)\begin{pmatrix} e^{i\alpha_{t,k}} & ie^{\omega_j^i(\lambda)}e^{i\beta_{t,k}} \\ ie^{\omega_j^i(\lambda)}e^{-i\beta_{t,k}} & e^{-i\alpha_{t,k}} \end{pmatrix} \qquad \alpha_{t,k}, \beta_{t,k} \in \mathbb{R}, \lambda \in \mathbb{C} \tag{1.2.20}$$

*where* $e^{\omega_j^i(\lambda)} = e^{(-1)^i\lambda+\omega_j^i} = e^{(-1)^i\lambda}\tan\frac{\delta_j^i}{2}$ *and* $\cos\delta_j^i(\lambda) = \frac{1-e^{2\omega_j^i(\lambda)}}{1+e^{2\omega_j^i(\lambda)}}$, $\delta_j^i$ *as before, which satisfies the compatibility condition ($k-t$ even)*

$$T_{t+1,k}(\lambda)T_{t+1,k-1}(\lambda) = T_{t,k}(\lambda)T_{t,k-1}(\lambda) \qquad \lambda \in \mathbb{C} \tag{1.2.21}$$

*defines a lorentzharmonic Gaussmap in a discrete (weak) Chebychev parametrization (1.2.11, 1.2.10 via*

$$N_{t,k} := -i\phi_{t,k}^{-1}\sigma_3\phi_{t,k}, \tag{1.2.22}$$

*where $\phi$ is a solution to the initial value ($k-t$ even) problem*

$$\phi_{t,k} := T_{t,k}\phi_{t-1,k-1} = T_{t+1,k}\phi_{t+1,k-1} \qquad \phi_{0,0} \in SU(2).$$

**Proof:** In [7] it was proven that if $T_{t,k}$ is of the form (1.2.20) but subject to the gauge constraints

$$\begin{array}{rrll} \alpha_{t,k} = h_{t,k} - h_{t-1,k-1} & \beta_{t,k} = 0 & \text{for} & k-t \quad \text{even} \\ \alpha_{t,k} = 0 & \beta_{t,k} = h_{t-1,k} + h_{t,k-1} & \text{for} & k-t \quad \text{odd} \end{array} \tag{1.2.23}$$

satisfies the zerocurvature condition (1.2.21) then this defines via the above mentioned definitions for all values of $\lambda$ a discrete lorentzharmonic Gaussmap in a (weak) Chebychev parametrization.

The normals are invariant under the gauge transformations

$$\begin{array}{lllllll} T_{t,k} & \rightarrow & \tilde{T}_{t,k} & := & A_{t,k}T_{t,k}A_{t-1,k-1}^{-1} & \text{for} & k-t \quad \text{even} \\ T_{t,k} & \rightarrow & \tilde{T}_{t,k} & := & A_{t-1,k}T_{t,k}A_{t,k-1}^{-1} & \text{for} & k-t \quad \text{odd} \end{array} \tag{1.2.24}$$

where

$$A_{t,k} := e^{ia_{t,k}\sigma_3} \tag{1.2.25}$$

For the case $\lambda = 0$ it was shown in [7] that if we do not fix any boundary conditions then any configuration of $\tilde{T}_{t,k}(0)$ can be gauged with the above $A_{t,k}$ into $T_{t,k}(0)$ of (1.2.23). On the other hand one sees immediately that the above gauge transformation doesn't change the form (1.2.20)of the $\tilde{T}_{t,k}$. Hence it follows that any configuration of $\tilde{T}_{t,k}(\lambda)$ can be gauged with $A_{t,k}$ (1.2.25) into $T_{t,k}(\lambda)$ of (1.2.23). □

By the above it became clear why for a given initial configuration of transport matrices $(\ldots T_{0,-1}, T_{0,0}, T_{0,1} \ldots)$ the zerocurvature condition (1.2.21) won't be sufficient for defining an evolution, unless we define additional gauge fixing constraints.

In the following sections we will investigate this option and moreover explain how some features of the above evolution can be viewed in terms of symplectic geometry. For this purpose it will be convenient to regauge the frame $\phi_{t,k}$ along every second diagonal of the lattice.

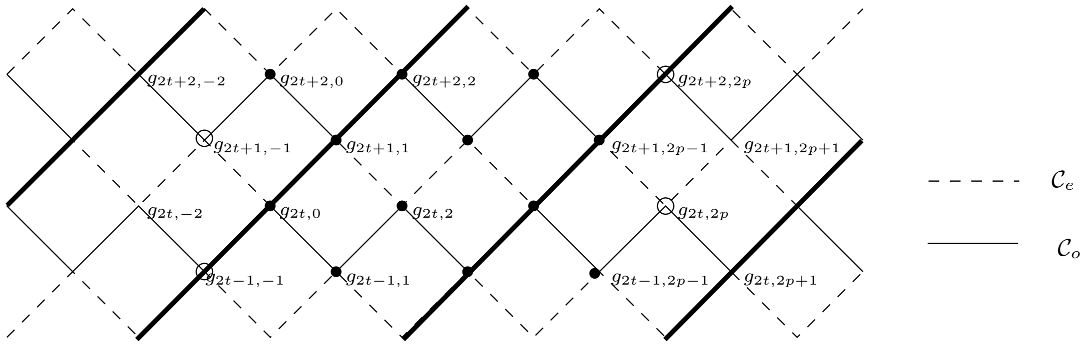
With respect to the so new obtained frame the transport matrices $T_{t,k}$ will assume a form, which shows that the above decribed model is more or less equivalent to the socalled Volterra model, appearing in the physics literature (see e.g. [54]).

## 1.3 Turning normals along every second diagonal

In the following we will consider a frame, which will be rotated along every second diagonal of the lattice by an angle of $\pi$ around the $x$-axes. If the variables $\phi_{t,k}$ obey some periodicity conditions we assume that the halfperiod $p$ of the lattice is an even number.Define

$$D_1(\phi_{t,k}) := \left\{ \begin{array}{ccc} \phi_{t,k} & \frac{k-t}{4} & \text{integer} \\ \sigma_1\phi_{t,k} & \frac{k-t}{4} & \text{noninteger} \end{array} \right.$$

$$D_2(\phi_{t,k}) := \left\{ \begin{array}{ccc} \sigma_1\phi_{t,k} & \frac{k-t}{4} & \text{integer} \\ \phi_{t,k} & \frac{k-t}{4} & \text{noninteger} \end{array} \right.$$




The transformations $D_1$ and $D_2$ differ only by a global regauge in $\sigma_1$ so that in the forthcoming it will be sufficient to study transformation $D_1$ only. The transport matrices with respect to $D_1(\phi_{t,k})$ read as:
$k-t$ even

$$\begin{array}{rcll} D_1(T_{t,k}) & := & D_1(\phi_{t,k})D_1(\phi_{t-1,k-1})^{-1} & \\ & = & T_{t,k} & \text{if } \frac{k-t}{4} \quad \text{integer} \\ & = & \sigma_1 T_{t,k}\sigma_1 & \text{if } \frac{k-t}{4} \quad \text{noninteger} \end{array}$$

$k-t$ odd

$$\begin{array}{rcll} D_1(T_{t,k}) & = & T_{t,k}\sigma_1 & \text{if } \frac{k-t+1}{4} \quad \text{integer} \\ & = & \sigma_1 T_{t,k} & \text{if } \frac{k-t+1}{4} \quad \text{noninteger} \end{array}$$

The swapped normal vectors differ only by signs,i.e

$$N_{t,k} = \left\{ \begin{array}{lll} D_1(N_{t,k}) := -iD_1(\phi_{t,k})^{-1}\sigma_3 D_1(\phi_{t,k}) & \frac{k-t}{4} & \text{integer} \\ -D_1(N_{t,k}) := +iD_1(\phi_{t,k})^{-1}\sigma_3 D_1(\phi_{t,k}) & \frac{k-t}{4} & \text{noninteger} \end{array} \right.$$

Hence if $A_{t,k} = e^{i\alpha_{t,k}\sigma_3} \in SU(2)$ then $N_{t,k}$ is invariant under the gauge transformation $D_1(\phi) \to A_{t,k}D_1(\phi_{t,k})$. It follows that instead of the space of all solutions $T_{t,k}$ to the zero curvature condition (for a given initial zig-zag of matrices $(\ldots T_{0,1}, T_{0,0}, T_{0,1}\ldots)$ which are subject to the gauge freedom 1.2.24 we can equally investigate the space of all solutions $D_1(T_{t,k})$ subject to the same gauge freedom.

Let us define:
$k-t$ even

$$\begin{array}{lll} u_{t,k} := \alpha_{t,k} & v_{t,k} = -\beta_{t,k} + \frac{\pi}{2} & \text{if } \frac{k-t}{4} \quad \text{integer} \\ u_{t,k} := -\alpha_{t,k} & v_{t,k} = +\beta_{t,k} + \frac{\pi}{2} & \text{if } \frac{k-t}{4} \quad \text{noninteger} \end{array}$$

$k-t$ odd

$$\begin{array}{lll} u_{t,k} := \beta_{t,k} & v_{t,k} = -\alpha_{t,k} - \frac{\pi}{2} & \text{if } \frac{k-t}{4} \quad \text{integer} \\ u_{t,k} := -\beta_{t,k} & v_{t,k} = +\alpha_{t,k} - \frac{\pi}{2} & \text{if } \frac{k-t}{4} \quad \text{noninteger} \end{array} \tag{1.3.26}$$

then for $k-t$ even

$$D_1(T_{t,k}) = \frac{1}{\cos\frac{\delta^2_{k+t}(\lambda)}{2}} \begin{pmatrix} e^{iu_{t,k}} & -e^{\lambda+\omega^2_{k+t}}e^{-iv_{t,k}} \\ e^{\lambda+\omega^2_{k+t}}e^{iv_{t,k}} & e^{-iu_{t,k}} \end{pmatrix}$$

$k-t$ odd

$$D_1(T_{t,k}) = \frac{i\sin\frac{\delta^1_{k-t}(\lambda)}{2}}{(\cos\frac{\delta^1_{k-t}(\lambda)}{2})^2}\begin{pmatrix} e^{iu_{t,k}} & -e^{\lambda-\omega^1_{k-t}}e^{-iv_{t,k}} \\ e^{\lambda-\omega^1_{k-t}}e^{iv_{t,k}} & e^{-iu_{t,k}} \end{pmatrix}$$

The swapped normal vectors differ only by signs,i.e

$$N_{t,k} = \begin{cases} D_1(N_{t,k}) := -iD_1(\phi_{t,k})^{-1}\sigma_3 D_1(\phi_{t,k}) & \frac{k-t}{4} \quad \text{integer} \\ -D_1(N_{t,k}) := +iD_1(\phi_{t,k})^{-1}\sigma_3 D_1(\phi_{t,k}) & \frac{k-t}{4} \quad \text{noninteger} \end{cases}$$

Hence if $A_{t,k} = e^{i\alpha_{t,k}\sigma_3} \in SU(2)$ then $N_{t,k}$ is invariant under the gauge transformation $\phi \to A_{t,k}D_1(\phi_{t,k})$. It follows that instead of the space of all solutions $T_{t,k}$ to the zero curvature condition (for a given initial zig-zag of matrices $(\ldots T_{0,1}, T_{0,0}, T_{0,1}\ldots)$ which are subject to the gauge freedom 1.2.24 we can equally investigate the space of all solutions $D_1(T_{t,k})$ subject to the same gauge freedom.

The factors in front of the matrices $D_1(T_{t,k})$ are unimportant for the investigation of the above described solution space. We will investigate from now on the solution space of the zero curvature condition:

$$L_{t+1,k}L_{t+1,k-1} = L_{t,k}L_{t,k-1} \tag{1.3.27}$$

where

$$L_{t,k}(\lambda + (-1)^i\omega^i_j) := \begin{pmatrix} e^{iu_{t,k}} & -e^{\lambda+(-1)^i\omega^i_j}e^{-iv_{t,k}} \\ e^{\lambda+(-1)^i\omega^i_j}e^{iv_{t,k}} & e^{-iu_{t,k}} \end{pmatrix} \tag{1.3.28}$$

and $\omega^i_j$ as before.

# Chapter 2

# Classical dynamics

After a brief review of essential definitions within symplectic geometry we will introduce the notion of discrete space time and dynamics on it, furthermore we will introduce an action, which is defined in terms of quasi periodic field variables. These field variables may be envisaged as "living" on the vertices of the Minkowski space time lattice ("vertex variables").

The Euler-Lagrange equations, which are obtained from this action by a variation in the (vertex) field variables are (modulo a redefinition) of Hirota and Sine-Gordon type.

The introduction of field variables, which "live" on the edges of the Minkowski space time lattice ("edge variables")leads to a better understanding of the role of the vertex fields variables.

## 2.1 Review of definitions within symplectic geometry

**Definition 2.1.1** *A* **Poisson structure** *on a M is a bilinear map on the algebra of $C^\infty$-functions on M*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

*satisfying the following conditions, for $f, g, h \in C^\infty(M)$:*

1. *Skewsymmetry*
$$\{f, g\} = -\{f, g\}$$

2. *Numbercommutativity*
$$\{f, c\} = 0 \quad if \quad c = const|_{\mathcal{M}}$$

3. *Leibnitz rule*
$$\{f, gh\} = g\{f, h\} + h\{f, g\}$$

4. *Jacobi identity*

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{f, h\}\} = 0$$

Let $(U_i, q_i)_{i\in J}$ be local charts on a smooth manifold M of dim $n$, then

$$\{f, g\}|_{U_i} = \sum_{l,m=0}^{n-1} \eta^{lm} \frac{\partial f}{\partial q_i^l} \frac{\partial g}{\partial q_i^m}$$

where $\eta$ is a rank two antisymmetric contravariant tensor which is defined by the conditions of definition 2.1.1. In particular $\eta$ transforms such that the definition of the Poisson bracket $\{\cdot, \cdot\}$ is independent of the choice of the coordinate maps $q_i^k : \mathcal{M} \to \mathbb{R}$

**Lemma 2.1.2** *If $f \in C^\infty(M)$, $\eta$ nondegenerate, $U_i \subset \mathcal{M}$ as before and*

$$\{q_i^k, f\} = 0$$

*for any chart components $q_i^k$ on $U_i \subset \mathcal{M}$ then $f = const.$ on $U_i$.*

**Proof:**
Since $\eta$ is nondegenerate the vectors $v_k$ defined by

$$(v_k)^m = \eta^{km}$$

are linearily independent f. a. $p \in U_i$, hence if equation

$$\{q_i^k, f\} = \sum_{l,m=0}^{n-1} \eta^{lm} \frac{\partial q_i^k}{\partial q_i^l} \frac{\partial f}{\partial q_i^m} = \sum_m^{n-1} (v_k)^m \frac{\partial f}{\partial q_i^m} = 0$$

holds for all $v_k$, $k \in \{0...n-1\}$ we obtain $\frac{\partial f}{\partial q_i^m} = 0$ (pointwise). Since $U_i$ is per assumption simply connected the assertion follows. □
If $\eta$ is nondegenerate for all $p \in U_i$ one can establish an isomorphism between the poisson structure $\{,\}$ on $\mathcal{M}$ and a bilinear closed differential form $\omega$ via the following: Let $\{dq_i^j\}$ be a canonical basis on the cotangent bundle $T^*\mathcal{M}$ restricted to the neighbourhood $U_i$ , such that

$$dq_i^j(\frac{\partial}{\partial q_i^k}) = \delta_{jk}, \qquad \text{on} \quad U_i$$

Let $f \in C^\infty(M)$ and

**Definition 2.1.3** *$X_f$ be the socalled* **hamiltonian vector field** *which is defined by $dq_i^l(X_f)|_p = \sum_{m=0}^{n-1} \eta^{lm}(q) \frac{\partial f}{\partial q_i^m}$;*

$\eta$ as before defined in definition 2.1.1. Then $\omega$ is the two form which maps $X_f$ into the total differential of $f$ : $\omega(X_f, \cdot) = df$, i.e. again in local coordinates:

$$\omega = \sum_{l,m=0}^{n-1} \eta_{lm} dq_i^m dq_i^l,$$

where $\eta_{lm}$ denotes the inverse of $\eta^{lm}$, i.e.

$$\sum_m \eta_{lm}\eta^{mk} = \delta_l^k.$$

**Definition 2.1.4** *A nondegenerate closed skewsymmetric two form $\Omega$ on a smooth manifold $\mathcal{M}$ is called a* **symplectic twoform***.*

*A manifold $\mathcal{M}$ which is eqipped with a symplectic twoform $\Omega$ is called a* **symplectic** *manifold $(\mathcal{M}, \Omega)$.*

Hence the above defined $\omega$ is a symplectic twoform. Sometimes $\Omega$ is degenerate on the manifold $\mathcal{M}$. In this case $\mathcal{M}$ has to be reduced to a submanifold in order to find a nondegenerate two form, which is invertible everywhere and induces a Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{M}$ (symplectic reduction).

An example of such a reduction is the socalled **Marsden-Weinstein reduction**, which will be explained soon.

**Definition 2.1.5** *An action of a Lie group $\mathcal{G}$ on $\mathcal{M}$ , is a smooth mapping $\phi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that f.a. $x \in \mathcal{M}$; $\quad g, h \in \mathcal{G}$*

a. *$\phi(e, x) = x$, where $e$ is the identity in $\mathcal{G}$*

b. *$\phi(g, \phi(h, x)) = \phi(gh, x)$.*

*Let $\phi_g := \phi(g, \cdot)$.*
*An action $\phi$ is called* **free** *iff the map $g \mapsto \phi_g$ is one to one for all $g \in \mathcal{G}$*
*An action $\phi$ is called* **proper** *iff $\tilde{\phi} : \mathcal{G} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ defined by $\tilde{\phi}(g, x) := (x, \phi(g, x))$ is a proper mapping, that is, if $K \subset \mathcal{M} \times \mathcal{M}$ is compact then $\tilde{\phi}^{-1}(K)$ is compact.*
*The map $\phi_g : \mathcal{M} \to \mathcal{M}$ is called* **symplectic** *iff the symplectic form $\omega$ on $\mathcal{M}$ is invariant under the pullback via the map $\phi_g$:*

$$\phi_g^* \omega = \omega$$

*An action $\phi$ is called* **symplectic** *iff the above defined $\phi_g$ is symplectic for all $g \in \mathcal{G}$.*

**Proposition 2.1.6** *[45] If $\phi : G \times \mathcal{M} \to \mathcal{M}$ is a proper free smooth action then $\mathcal{M} \backslash \mathcal{G}$is a smooth manifold and $\pi : \mathcal{M} \to \mathcal{M} \backslash \mathcal{G}$ is a submersion.*

**Definition 2.1.7** *An infinitesimal generator $\xi^M$ of a smooth action $\phi$ is a vector field $\xi^M \in \mathcal{X}(\mathcal{M})$ defined by:*

$$\xi^M|_p := \frac{d}{dt}|_{t=0}\,\phi(exp(t\xi), p),$$

*where $\xi \in \mathfrak{g}$(the Liealgebra of $\mathcal{G}$). The map $\phi(exp(t\xi), \cdot) : (-\epsilon, \epsilon) \times \mathcal{M} \to \mathcal{M}$ will be called a $\xi$-**flow** on $\mathcal{M}$.*

**Definition 2.1.8** *A **momentum function** corresponding to a symplectic flow $\phi_{exp(t\xi)}$ on a smooth connected manifold is a smooth function*

$$\hat{J}(\xi) : \mathcal{M} \to \mathbb{R}$$

*such that the infinitesimal generator $\xi^M \in \mathcal{X}(\mathcal{M})$ belonging to $\phi_{exp(t\xi)}$ is a hamiltonian vector field, i.e.*

$$X_{\hat{J}(\xi)}|_p = \xi^M|_p.$$

**Definition 2.1.9** *Let $\hat{J}(\xi)$ defined as above be defined for all $\xi \in g$. The **momentum mapping** $J$ for a symplectic action $\phi$ on a smooth manifold $\mathcal{M}$ is the map $J : \mathcal{M} \to \mathfrak{g}^*$ (the dual of the Liealgebra) defined by*

$$J(p)(\xi) := \hat{J}(\xi)(p)$$

**Definition 2.1.10** *Let $I(h) : \mathcal{G} \to \mathcal{G}$ be the inner automorphism on $\mathcal{G}$ associated with $h$ which is defined by*

$$I(h)(g) = hgh^{-1}$$

*The map $I(h)$ induces an automorphism $Ad_h : \mathfrak{g} \to \mathfrak{g}$ on the Liealgebra $\mathfrak{g}$, namely if $\xi_g \in \mathfrak{g}$ denotes the Liealgebra element corresponding to $g$ then*

$$\xi_g \mapsto Ad_h(\xi_g) = \xi_{hgh^{-1}}.$$

*$Ad_h$ is called the **adjoint mapping** associated with $h$.*

Let us consider the function $\psi_{g,\xi} : \mathcal{M} \to K$ defined by:

$$p \mapsto \hat{J}(\xi)(\phi_h(p)) - \hat{J}(Ad_{h^{-1}}\xi)(p)$$

It is straightforward to show that $d\psi_{h,\xi} = 0$. Hence $\psi_{h,\xi}$ is constant on a simply connected neighbourhood $U_i$. $\psi_{h,\xi}$ somewhat measures how the commutator (i.e. the Poisson bracket) between the momentum functions $\{\hat{J}(\xi_g), \hat{J}(\xi_h)\}$ translates into the commutator in the Liealgebra. If we demand that $\psi_{h,\xi}|_{\mathcal{M}} = 0$ then we restrict ourselves to certain moment maps, which are called $Ad^*$-equivariant:

**Definition 2.1.11** *A momentum mapping $J$ is called $Ad^*$-**equivariant** provided*

$$J(\phi_h(x)) = Ad^*_{h^{-1}} J(x)$$

It is now straightforward to find that if $J$ is $Ad^*$-equivariant, then

$$\{\hat{J}(\xi_g), \hat{J}(\xi_h)\} = \hat{J}([\xi, \eta]). \tag{2.1.1}$$

For $\mu \in \mathfrak{g}^*$ let $G_\mu = \{g \in \mathcal{G} | Ad^*_{g^{-1}}\mu = \mu; \mu \in \mathfrak{g}^*\}$ be the isotropy group under the $Ad^*$-action on $\mathfrak{g}$. Since $J$ is $Ad^*$-equivariant we have for $g_\mu \in \mathcal{G}_\mu$

$$J(\phi_{g_\mu}(J^{-1}(\mu))) = Ad^*_{g_\mu^{-1}}\mu = \mu \tag{2.1.2}$$

hence

$$\phi_{g_\mu}(J^{-1}(\mu)) \subset J^{-1}(\mu).$$

Since the action $\phi$ is per assumption proper and free for all $g \in \mathcal{G}$ and since $\phi_{G_\mu}$ leaves $J^{-1}(\mu)$ for any regular value $\mu \in \mathfrak{g}^*$ invariant, we know by proposition 2.1.6 that the space of all equivalence classes via $\phi_{g_\mu} x \equiv x$, i.e. the quotient

$$J^{-1}(\mu)/G_\mu$$

is well defined. It is called **reduced phase space** and the reduction as explained above is called **Marsden-Weinstein reduction**.
The following example will give a simple application of the above construction of $P_\mu = J^{-1}(\mu)/\mathcal{G}_\mu$ in addition it will serve as a proof of an assertion to come up later.

### 2.1.1 Example

Let $\mathcal{M} = \mathbb{R}^{2p+2}$, let $\{g_i\}_{i\in\{0\dots 2p+1\}}$ be canonical coordinates on $\mathcal{M}$.
Define an action $\phi$ of the Liegroup $\mathcal{G} = \mathbb{R} \times \mathbb{R}$ by

$$\begin{array}{rccl} \phi \; : & \mathcal{G} \times \mathcal{M} & \to & \mathcal{M} \\ & ((t_1,t_2), g = \sum_{j=0}^{2p+1} g_j e_j) & \mapsto & \phi((t_1,t_2),g) = \\ & & & \sum_{j=0}^{p}(g_{2j}+t_1)e_{2j} + \sum_{j=0}^{p}(g_{2j+1}+t_2)e_{2j+1} \end{array}$$

where $\{e_i\}_{i\in\{0\dots 2n-1\}}$ is the standard euclidean basis on $\mathbb{R}^{2p+2}$, i.e. $dg_j(e_k) = \delta_{jk}$.

Clearly $\phi$ is proper and transitive. The infinitesimal generators corresponding to the canonical basis in $\mathfrak{g}$ are

$$\xi^M_{t_1} = \frac{d}{dt}|_0 \phi((t_1 t, 0), p) = \sum_{j=0}^{p} e_{2j} t_1$$

$$\xi_{t_2}^M = \frac{d}{dt}|_0 \phi((0, t_2 t), p) = \sum_{j=0}^{p} e_{2j+1} t_2$$

as the exponential map $\mathfrak{g} \to \mathcal{G}$ is the identity map on $\mathbb{R} \times \mathbb{R}$. Now let us make use of the following symplectic form $\Omega$ on $\mathbb{R}^{2p+2}$ namely:

$$\begin{aligned} \omega = -\frac{1}{2}\Big( & \sum_{k=0}^{p-1} \mathbf{d}g_{2k+1} \wedge (\mathbf{d}g_{2k} - \mathbf{d}g_{2k+2}) \\ & + (\mathbf{d}g_{2p-1} - \mathbf{d}g_{-1}) \wedge \mathbf{d}g_0 + \frac{1}{2}(\mathbf{d}g_{2p-1} - \mathbf{d}g_{-1}) \wedge (\mathbf{d}g_{2p} - \mathbf{d}g_0)\Big) \end{aligned}$$

Define $\hat{J}(\xi_{t_i}) : \mathcal{M} \to \mathbb{R}$ by

$$\hat{J}(\xi_{t_1}) = t_1 \frac{1}{2}(g_{2p-1} - g_{-1}) := \frac{1}{2} m^{(2)} t_1 \qquad \hat{J}(\xi_{t_2}) = t_2 \frac{1}{2}(g_{2p} - g_0) := \frac{1}{2} m^{(1)} t_2.$$

Hence

$$d\hat{J}(\xi_{t_1}) = \frac{1}{2}(dg_{2p-1} - dg_{-1}) t_1 \qquad d\hat{J}(\xi_{t_2}) = \frac{1}{2}(dg_{2p} - dg_0) t_2.$$

Now

$$\begin{aligned} \Omega(\xi_{t_1}^M, \cdot) &= t_1 \frac{1}{2}(dg_{2p-1} - dg_{-1}) = d\hat{J}(\xi_{t_1}) \\ \Omega(\xi_{t_2}^M, \cdot) &= -t_2 \frac{1}{2} \sum_{k=0}^{p-1} dg_{2k} - dg_{2k+2} = t_2 \frac{1}{2}(dg_{2p} - dg_0) \\ &= d\hat{J}(\xi_{t_2}). \end{aligned}$$

So $\hat{J}(\xi_{t_i})$ defines the momentum mapping $J : \mathbb{R}^{2p+2} \to \mathfrak{g}^* = \mathbb{R} \times \mathbb{R}$

$$J = \frac{1}{2} m^{(2)} dt_1 + \frac{1}{2} m^{(1)} dt_2$$

where $dt_i$ denotes the dual basis in $\mathfrak{g}^*$. $J$ is trivially $Ad^*$-equivariant and the isotropy group $\mathcal{G}_{(\rho,\mu)} = \mathcal{G}$, for arbitrary $\rho, \mu \in \mathbb{R}$ i.e. we find for $p_0$ fixed that

$$\begin{aligned} \hat{J}(\xi_{t_1})(\phi((t_1, t_2), p_0) &= \rho \\ \hat{J}(\xi_{t_2})(\phi((t_1, t_2), p_0) &= \mu. \end{aligned}$$

Hence the reduced phase space

$$P_\mu = J^{-1}((\rho, \mu))/\mathcal{G}_{(\rho,\mu)}$$

is the submanifold of $\mathbb{R}^{2p+2}$, which is coordinatized by the difference variables $p_k$ which are invariant under the action of $\mathcal{G}$ and for which the functions $\hat{J}(\xi_{t_i}) = const$.

## 2.2 Discrete space time and time evolution

Space time is usually understoood as being a certain manifold which is supplied with a causal structure. The causal structure is provided by a metric distinguishing time, null- and spacelike distances together with the principle of local causality, which is that two events (points) in a convex neighbourhood $U$ are causally related if there exists in $U$ a nonspacelike curve connecting them. If one introduces matter fields then further conditions on the metric will be necessary. For the moment we will be concerned with vacuum solutions, i.e. no matter field solutions. Hence sofar we only have to implement the above principle of causality into a notion of a discrete space time.

In our (not necessarily compact) discrete space time, causality will be implemented by considering ”nonspacelike curves” (directed paths), where two events (sitting on vertices) are causally related, if there exists a path between them.

**Definition 2.2.1** *A* **directed graph** *is a tupel $\mathcal{G} = (V, E)$ of a set $V$ (vertices) and ordered pairs $E = V \times V$ (edges) .*
*Define for a pair $(v_1, v_2) \in E$ , $s(e) := v_1$ (source), $t(e) := v_2$ (target).*
*A* **path** *in a directed graph $\mathcal{G} = (V, E)$ is a (not necessarily finite) sequence $\gamma = (\ldots e_0, e_1, e_2 \ldots)$ of edges $e_i \in E$, such that $t(e_i) = s(e_{i+1})$, $i \in \mathbb{N}$. Denote the set of all paths with $\Gamma$.*
*An* **undirected path** *in a directed graph $\mathcal{G} = (V, E)$ is a (not necessarily finite) sequence $\gamma = (\ldots e_0, e_1, e_2 \ldots)$ of undirected edges, i.e. unordered pairs $e = \{v_1, v_2\}$ such that if $e_i = \{v_i, v_{i+1}\} \in unordered(E)$ then $(v_i \in e_{i-1} \wedge v_{i+1} \in e_{i+1}) \vee (v_{i+1} \in e_{i-1} \wedge v_i \in e_{i+1})$. Denote the set of all undirected paths with $U\Gamma$.*
*A* **cycle** *is a finite path $(e_0, \ldots, e_N)$ such that $e_0 = e_N$.*
*Two paths $\gamma_1, \gamma_2 \in \Gamma$* **cut** *themselves* **once** *iff*

$$\gamma_1 \cap \gamma_2 \in \Gamma \wedge \#\{e \in E | e \in \gamma_1 \cap \gamma_2\} < \infty.$$

*Analogously we define a cut also between undirected and undirected with directed paths.*

**Lemma 2.2.2** *If there are no nontrivial cycles , then the relation $R$ given by*

$$e_1 R e_2 \Leftrightarrow \exists \quad \gamma \in \Gamma \text{ with } \gamma = (e_1 \ldots e_2)$$

*defines a half-ordering on $E$ .*

**Proof:** Reflexivity and transitivity holds trivially, antisymmetry ($aRb \wedge bRa \Rightarrow a = b$) is fulfilled since there are no nontrivial cycles.

Denote $R$ with $\leq$.

**Definition 2.2.3** *A* **discrete spacetime** *$\mathcal{S}$ is a directed graph $\mathcal{G} = (V, E)$ with the below properties.*

a.) *There are no nontrivial cycles*

b.) *For all* $v \in V$ *the number of incoming edges and outgoing edges is finite.*

c.) *Let* $Past(e) = \{\tilde{e} \in M | \tilde{e} \leq e\}$, $Future(e) = \{\tilde{e} \in M | \tilde{e} \geq e\}$ *then* $\# Past(e) \cap Future(\tilde{e}) = \#\{\hat{e} \in M | \tilde{e} \leq \hat{e} \leq e\} < \infty$

The spacetime we are going to consider will be twodimensional so the following definitions will become important.

**Definition 2.2.4** *Let* $\gamma_A \in U\Gamma$ *be an undirected path.* $\gamma_A \in U\Gamma$ *is called* **acausal** *iff*

$$\exists \, no \; path \, \gamma \in \Gamma \; which \; cuts \, \gamma_A \; more \; than \; once.$$

*Denote the set of all acausal paths with* $A(\Gamma)$.
*A* **Cauchy path** $\mathcal{C} \in A(\Gamma)$ *is an acausal undirected path for which the following holds* $\forall e \in E$:

$$e \cup \mathcal{C} \in A(\Gamma) \Rightarrow e \in \mathcal{C}$$

*The set of all Cauchy paths will be denoted with* $\Gamma_{\mathcal{C}}$.

**Definition 2.2.5** *A Cauchy path* $\mathcal{C}_a$ *is* **later** *than a Cauchy path* $\mathcal{C}_b$ , $\mathcal{C}_a \neq \mathcal{C}_b$ *if for all paths* $\gamma$

$$\gamma \cap \mathcal{C}_a \geq \gamma \cap \mathcal{C}_b. \tag{2.2.3}$$

Note that the existence of a Cauchy path is a rather strong requirement.
We are now in the position to define an evolution for field variables.
Depending on the considered models, let $q^{\mathcal{C}}$ be a function on primitives $a_i$ like edges or vertices along a Cauchy path $\mathcal{C}$. Let $n = \#\{a_i | a_i \; along \; \mathcal{C}\}$, up to now $n$ might be still infinite. We write shortly $q^{\mathcal{C}} : \mathcal{C} \to \mathbb{R}^n$ or $q^{\mathcal{C}} \in \mathcal{F}(\mathcal{C}, \mathbb{R}^n)$ instead of $q^{\mathcal{C}} : Primitives(\mathcal{C}) \to \mathbb{R}^n$. Denote $q^{\mathcal{C}}_i := q^{\mathcal{C}}(a_i)$ where $a_i$ denotes the corresponding primitive.

**Definition 2.2.6** *A* **time evolution** *is a bijective map* $T_{\mathcal{C}_2,\mathcal{C}_1} : \mathcal{F}(\mathcal{C}_1, \mathbb{R})^n \to \mathcal{F}(\mathcal{C}_2, \mathbb{R})^n$:

$$T_{\mathcal{C}_2,\mathcal{C}_1} : \begin{pmatrix} q_0^{\mathcal{C}_1} \\ . \\ . \\ q_{n-1}^{\mathcal{C}_1} \end{pmatrix} \mapsto \begin{pmatrix} q_0^{\mathcal{C}_2} \\ . \\ . \\ q_{n-1}^{\mathcal{C}_2} \end{pmatrix} \tag{2.2.4}$$

*where* $\mathcal{C}_2$ *is later than* $\mathcal{C}_1$.
*Define* $T_{\mathcal{C}_N,\ldots,\mathcal{C}_0} := T_{\mathcal{C}_N,\mathcal{C}_{N-1}} \ldots T_{\mathcal{C}_2,\mathcal{C}_1} T_{\mathcal{C}_1,\mathcal{C}_0}$.
*A time evolution is* **unique** *or* **flat** *iff* $T_{\mathcal{C}_N,\mathcal{C}_0} := T_{\mathcal{C}_N,\mathcal{C}_{N-1}} \ldots T_{\mathcal{C}_2,\mathcal{C}_1} T_{\mathcal{C}_1,\mathcal{C}_0}$ *is well defined for all* $n \in \mathbb{N}$ *, i.e. independent of the "path"* $\mathcal{C}_N, \ldots, \mathcal{C}_0$.

The sequence $q = (q_0^{\mathcal{C}_0} \dots q_{n-1}^{\mathcal{C}_0}, q_0^{\mathcal{C}_1} \dots q_{n-1}^{\mathcal{C}_1}, \dots, q_0^{\mathcal{C}_N} \dots q_{n-1}^{\mathcal{C}_N})$ will be called a **solution** to the time evolution $T_{\mathcal{C}_N,\dots,\mathcal{C}_0}$ for the initial values $q_0^{\mathcal{C}} = (q_0^{\mathcal{C}_0} \dots q_{n-1}^{\mathcal{C}_0})$ Hence the set of all solutions $\{q\}_{\mathcal{C}_N,\dots,\mathcal{C}_0}$ to (2.2.4), which will be called covariant phase space $\mathcal{M}$ is isomorphic to the set of all initial values $\{q^{\mathcal{C}_0}\}$. Furthermore the variables $q^{\mathcal{C}_0}$ can (while abusing notation) be viewed as local coordinates on $\mathcal{M}$ via

$$q_i^{\mathcal{C}_0}(q) = q_i^{\mathcal{C}_0} \in \mathbb{R}; \quad q \in \mathcal{M}; \quad i \in \{0, \dots, n-1\}$$

The set of all coordinates on $\mathcal{M}$ will be denoted as $Coord(\mathcal{M})$.

Since our evolution is invertible we could have however parametrized the solution $q \in \mathcal{M}$ by taking variables $q^{\mathcal{C}_j}, \quad j \neq 0$ on a e.g. later Cauchy path $\mathcal{C}_j$ as coordinates on phase space. The invertible transformation $\mathcal{T}_{\mathcal{C}_j,\mathcal{C}_k} : (C^\infty(\mathcal{M})) \to (C^\infty(\mathcal{M}))$ between any of these coordinates is then given by

$$\mathcal{T}_{\mathcal{C}_j,\mathcal{C}_k} \begin{pmatrix} q_0^{\mathcal{C}_k} \\ \cdot \\ \cdot \\ q_{n-1}^{\mathcal{C}_k} \end{pmatrix} (q) := T_{\mathcal{C}_j,\mathcal{C}_k} \begin{pmatrix} q_0^{\mathcal{C}_k} \\ \cdot \\ \cdot \\ q_{n-1}^{\mathcal{C}_k} \end{pmatrix}. \tag{2.2.5}$$

We assume from now on that $n < \infty$ and demand that $T_{\mathcal{C}_i,\mathcal{C}_k}$ is $C^\infty$ and canonically supply $\mathcal{M}$ with a $C^\infty$-structure via the above construction. For further use we remark that $T_{\mathcal{C}_i,\mathcal{C}_k}$ induces an evolution on the torus by:

$$\begin{aligned} R_{\mathcal{C}_j,\mathcal{C}_k} e^{iq^{\mathcal{C}_0}} &:= e^{iT_{\mathcal{C}_j,\mathcal{C}_k} q^{\mathcal{C}_0}} \\ \mathcal{R}_{\mathcal{C}_j,\mathcal{C}_k} e^{iq^{\mathcal{C}_0}} &:= e^{i\mathcal{T}_{\mathcal{C}_j,\mathcal{C}_k} q^{\mathcal{C}_0}} \end{aligned} \tag{2.2.6}$$

We note that if our phase space $\mathcal{M}$ carries a symplectic structure $\omega$, then $T_{\mathcal{C}_j,\mathcal{C}_k} = q^{\mathcal{C}_j} \circ (q^{\mathcal{C}_k})^{-1}$ is automatically a symplectic transformation, i.e.

$$((q^{\mathcal{C}_j} \circ id \circ (q^{\mathcal{C}_k})^{-1})^* q_*^{\mathcal{C}_j} \omega = q_*^{\mathcal{C}_k} \omega.$$

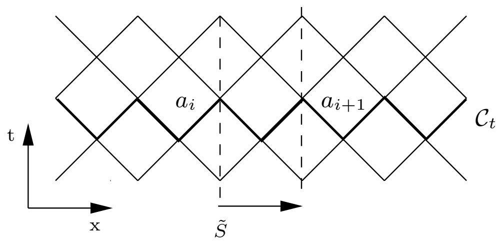
in other words $T_{\mathcal{C}_j,\mathcal{C}_k}$ is nothing else then the local representative of the identity map on $\mathcal{M}$. Reversing the construction one can define a meaningful symplectic structure on $\mathcal{M}$ by finding a nondegenerate twoform, written in local coordinates $q^{\mathcal{C}_k}$ and then showing its independence under the transformation $T_{\mathcal{C}_j,\mathcal{C}_k}$. This kind of construction will be used in later sections. Let us now define a bijection $\tilde{S}_{\mathcal{C}_k} : \mathcal{C}_k \to \mathcal{C}_k$ on the defining primitives of a Cauchypath $\mathcal{C}_k$. Pointwise we can define a map $S_{\mathcal{C}_k} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S_{\mathcal{C}_k}(q^{\mathcal{C}_k}(a_i)) := q^{\mathcal{C}_k}(\tilde{S}_{\mathcal{C}_k}(a_i)) \tag{2.2.7}$$

where $q^{\mathcal{C}_k}$ as above. Assume $S_{\mathcal{C}_k}$ is a diffeomorphism. Such a map will lift in general to a nontrivial diffeomorphism $\hat{S}_{\mathcal{C}_k} := (q^{\mathcal{C}_k})^{-1} \circ S_{\mathcal{C}_k} \circ q^{\mathcal{C}_k}$ on phase space. This diffeomorphism $\hat{S}_{\mathcal{C}_k}$ on phase space might nevertheless still symplectic, and so the corresponding newly obtained functions $\tilde{q}^{\mathcal{C}_k} := S_{\mathcal{C}_k} \circ q^{\mathcal{C}_k}$ could equally well serve as symplectic coordinates on phase space.

**Definition 2.2.7** *Let* $\tilde{S}_{\mathcal{C}_k} : \mathcal{C}_k \to \mathcal{C}_k$ *be a bijection on* $\mathcal{C}_k$*,* $q^{\mathcal{C}_k}$ *as above. Such a bijection* $\tilde{S}_{\mathcal{C}_k}$ *is called* **liftable** *to a diffeomorphism* $\hat{S}_{\mathcal{C}_k}$ *on* $\mathcal{M}$ *iff there exists a diffeomorphism* $S_{\mathcal{C}_k} : \mathbb{R}^n \to \mathbb{R}^n$ *for which equation (2.2.7) holds. Define* $\hat{S}_{\mathcal{C}_k} = (q^{\mathcal{C}_k})^{-1} \circ S_{\mathcal{C}_k} \circ q^{\mathcal{C}_k}$*.* $\tilde{S}_{\mathcal{C}_k}$ *is called* **symplectic liftable** *if* $\hat{S}_{\mathcal{C}_k}$ *is symplectic.*

We will turn our attention on models of right this type.
The space time lattice we consider will be from now on the Minkowski space time lattice depictured below, the considered evolutions will always be unique.




A canonical Cauchy $\mathcal{C}_t$ path will be the socalled Cauchy zig zag, as depictured in the figure above. The initial values along such a zig zag will obey some boundary conditions such as being periodic or quasiperiodic. Hence our phase space will be finite dimensional and can be supplied with a $C^\infty$ structure in the above way.

A bijection on such a Cauchy zig zag will be e.g. a shift $\tilde{S}$

$$\tilde{S}(a_i) = a_{i+1}.$$

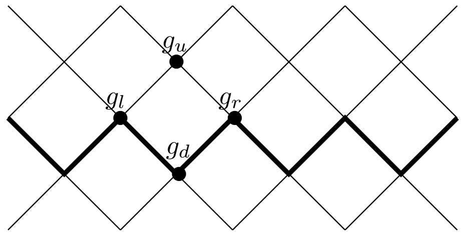
Clearly $\tilde{S}_{\mathcal{C}_k}$ is liftable to a diffeomorphism $\hat{S}_{\mathcal{C}_k}$ on $\mathcal{M}$. We now want to choose such a symplectic form on phase space, such that the diffeomorphism $\hat{S}$ on phase space will be a symplectic transformation. A symplectic form which is invariant under these transformations is henceforth called translational invariant symplectic form.

## 2.3 Dynamics of Hirota type

Let us investigate a class of integrable lattice systems (see e.g. [9, 43, 48, 49, 53]) defined by an evolution equation of the following type:

$$g_u - g_d - V'(g_l - g_r) = 0, \tag{2.3.8}$$

where $u$, $d$, $l$ and $r$ denote up, down, left and right, respectively and $V'(x)$ is the derivative of $V(x) : \mathbb{R} \to \mathbb{R}$. If we start with initial data on a Cauchy zig zag $\mathcal{C}_t$ on a light cone lattice (see figure below) the local evolution given by (2.3.8) will determine the function $g$ on a Cauchy path $\mathcal{C}_{t+1}$ and finally on the whole lattice.

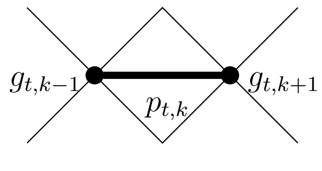



The map from the data on the Cauchy path $\mathcal{C}_t$ to the data on the Cauchy path $\mathcal{C}_{t+1}$:

$$T : \begin{pmatrix} g_0^t \\ \cdot \\ \cdot \\ g_{n-1}^t \end{pmatrix} \mapsto \begin{pmatrix} g_0^{t+1} \\ \cdot \\ \cdot \\ g_{n-1}^{t+1} \end{pmatrix} \tag{2.3.9}$$

is of the form (2.2.4) and so we will apply the in the previous section developped techniques.

Since the variables $g_{t,i}$ are labeled by their position on the vertices of the above space time lattice they will be briefly called "vertex variables". Correspondingly we will refer to the algebra generated by vertex variables along a Cauchy path as the "vertex algebra" . Equation (2.3.8) will be called of Hirotatype since one obtains the well known Hirota equation [29], which is equivalent to (2.3.8) for $V'(x) = -i \ln(\frac{1+ke^{ix}}{k+e^{ix}})$ if one redefines $u \to -u$ along every second diagonal of the light cone lattice (see also [20, 22]). The difference of two adjacent values of $g$, i.e. $p_{t,k} := g_{t,k-1} - g_{t,k+1}$ (see fig.1) for the potential above describes (modulo a redefinition along the diagonals [6]) the angle between the tangent directions of a discrete K-surface in the Tchebycheff parametrization [7]. Since the difference variables $p_{t,k}$ are functions of vertex variables which are adjacent in space they will also be called **face variables**.




A (spatially) periodic light cone lattice $L_{2p}$ with period $2p$ may be viewed as $L/\mathbb{Z}$, where $\mathbb{Z}$ acts on the infinite light cone lattice $L$ by shifts by $2p$ in space-like direction (cf. fig. 1). A quasi-periodic field is a mapping

$$g : L \longrightarrow \mathbb{R}$$

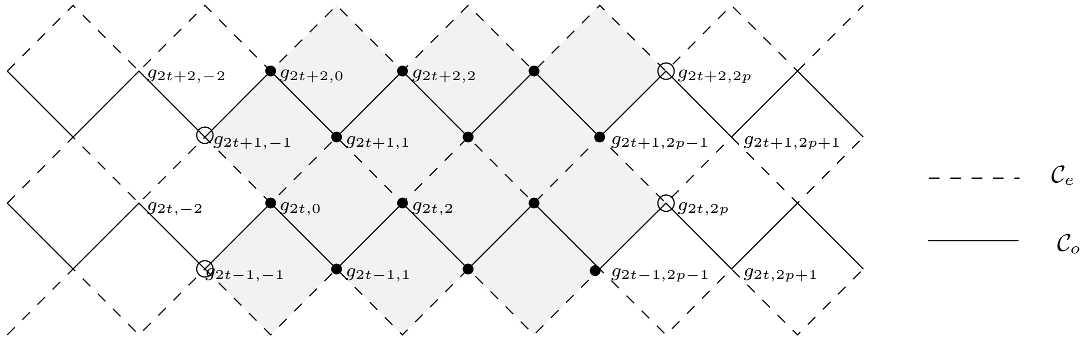
with

$$g_{t,i+2p} - g_{t,i} = g_{t,i+2p+2} - g_{t,i+2} \quad \forall i,$$

i.e., there are two (space independent) monodromies $m_t^{(1)}, m_t^{(2)}$ defined by

$$m_t^{(i)} = g_{t,2p+2k+1-i} - g_{t,2k+1-i} \tag{2.3.10}$$

for an arbitrary $k \in \mathbb{Z}$.




We denote by $\mathcal{P}$ the set of faces on a fundamental domain of the lattice (cf. fig. 1). On the set of quasi-periodic fields we define the following action:

$$\begin{aligned} S &= \sum_{P\in\mathcal{P}} \{g_d(P)(g_l(P)-g_r(P)) + V(g_l(P)-g_r(P))\} \\ &\quad + \sum_t \Big\{ g_{2t+1,2p-1} m^{(1)}_{2t} - \frac{1}{2} m^{(1)}_{2t} m^{(2)}_{2t+1} + g_{2t,0} m^{(2)}_{2t-1} + \frac{1}{2} m^{(1)}_{2t} m^{(2)}_{2t-1} \Big\} \qquad (2.3.11) \\ &= \sum_t \Big\{ \sum_{k=1}^{p-1} g_{2t,2k}(g_{2t+1,2k-1} - g_{2t+1,2k+1}) + g_{2t,0}(g_{2t+1,2p-1} - m^{(2)}_{2t+1} - g_{2t+1,1}) \\ &\quad + g_{2t+1,2p-1} m^{(1)}_{2t} - \frac{1}{2} m^{(1)}_{2t} m^{(2)}_{2t+1} \qquad (2.3.12) \\ &\quad + \sum_{k=0}^{p-2} g_{2t-1,2k+1}(g_{2t,2k} - g_{2t,2k+2}) + g_{2t-1,2p-1}(g_{2t,2p-2} - g_{2t,0} - m^{(1)}_{2t}) \\ &\quad + g_{2t,0} m^{(2)}_{2t-1} + \frac{1}{2} m^{(1)}_{2t} m^{(2)}_{2t-1} \\ &\quad + \sum_{k=1}^{p-1} V(g_{2t+1,2k-1} - g_{2t+1,2k+1}) + \sum_{k=0}^{p-2} V(g_{2t,2k} - g_{2t,2k+2}) \\ &\quad + V(g_{2t+1,2p-1} - m^{(2)}_{2t+1} - g_{2t+1,1}) + V(g_{2t,2p-2} - g_{2t,0} - m^{(1)}_{2t}) \Big\} \end{aligned}$$

This action differs from the action in [10], which is only defined on periodic fields, by the addition of the extra terms in the second line of equation (2.3.11). Those extra terms guarantee that the action is invariant under a space-like shift, and hence is independent of the choice of the fundamental domain. Note, that the monodromies entering the action may a priori be time dependent: They become dynamical variables rather than pure parameters, and their time independency will be a consequence of the equations of motion.

$\{(g_{t,k})_{t\in\mathbb{Z},k\in\{-1,\dots p-1\},k-t \text{ even}}\}$ and $\{(g_{t,k})_{t\in\mathbb{Z},k\in\{0,\dots p-1\},k-t \text{ even}}, m^{(1)}_{2t}, m^{(2)}_{2t+1}\}$ form independent coordinate systems on the set of quasi-periodic fields. Whereas the former is particularly useful for the derivation of the symplectic structure in the next section, the equations of motion may be most easily derived using the latter coordinate system.

Variation with respect to $g_{t,k}, (t \in \mathbb{Z}, k \in 0, \dots 2p-1)$, yields (exactly as in [10]) the difference of the evolution equations (2.3.8) for the neighbouring faces to the left and to the right of $g_{t,k}$:

$$\begin{aligned}
&g_{t,k} \text{ at even times:} \\
&g_{2t+1,2k-1} - g_{2t-1,2k-1} - g_{2t+1,2k+1} + g_{2t-1,2k+1} - V'(g_{2t,2k-2} - g_{2t,2k}) \\
&\qquad + V'(g_{2t,2k} - g_{2t,2k+2}) = 0, \\
&g_{t,k} \text{ at odd times:} \\
&g_{2t,2k} - g_{2t-2,2k} - g_{2t,2k+2} + g_{2t-2,2k+2} - V'(g_{2t-1,2k-1} - g_{2t-1,2k+1}) \\
&\qquad + V'(g_{2t-1,2k+1} - g_{2t-1,2k+3}) = 0
\end{aligned} \tag{2.3.13}$$

Variation with respect to the monodromies $m_t^{(i)}$, $i = 1, 2$ yields:

$$\begin{aligned}
g_{2t+1,2p-1} - g_{2t-1,2p-1} - V'(g_{2t,2p-2} - g_{2t,0} - m_{2t}^{(1)}) &= \tfrac{1}{2} m_{2t+1}^{(2)} - \tfrac{1}{2} m_{2t-1}^{(2)} \\
g_{2t+2,0} - g_{2t,0} - V'(g_{2t+1,2p-1} - m_{2t+1}^{(2)} - g_{2t+1,1}) &= \tfrac{1}{2} m_{2t}^{(1)} - \tfrac{1}{2} m_{2t+2}^{(1)}.
\end{aligned} \tag{2.3.14}$$

Now, as we may write the monodromy as the sum of differences of field variables:

$$m_t^{(i)} = \sum_{k=1}^{p} (g_{t,2k+1-i} - g_{t,2k-1-i})$$

equations (2.3.13) are sufficient to enforce the time independence of the monodromies. Hence, the right hand sides of equations (2.3.14) vanish. We get the evolution equations (2.3.8) for the faces "above" $g_{2t-1,2p-1}$ and $g_{2t,0}$ for all $t$, and thus finally for all faces.

### 2.3.1 Symplectic Structure on (vertex) covariant phase space

Using straightforward modifications of covariant phase space techniques [58, 13], we first show how one can derive a symplectic structure from any action on a light cone lattice which may be written as the sum of terms whith support on the "canonical Cauchy paths" $\mathcal{C}_{o,e}^t$ (cf. fig. 2). This symplectic structure will automatically be invariant under evolution.

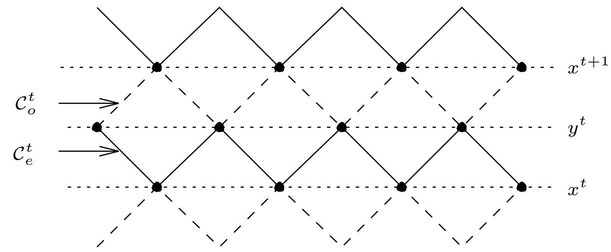



Figure 2

There is in general no canonical way of identifying $\mathcal{C}_o^t$ with $\mathcal{C}_e^t$. Therefore, our ansatz for the action is:

$$S = \sum_t \Big( L_1(\mathbf{x}^t, \mathbf{y}^t) + L_2(\mathbf{y}^t, \mathbf{x}^{t+1}) \Big)$$

with two different functions $L_1$ and $L_2$. Here, $\mathbf{x}^t$ are the variables on the lower vertices of $\mathcal{C}_e^t$, and $\mathbf{y}^t$ the variables on the upper vertices of $\mathcal{C}_e^t$ (or, equivalently, the lower vertices of $\mathcal{C}_o^t$).

Variation of the action with respect to $\mathbf{x}^t$ and $\mathbf{y}^t$ yields the equations of motion:

$$\begin{aligned} B_1^t &:= D_1 L_1(\mathbf{x}^t, \mathbf{y}^t) + D_2 L_2(\mathbf{y}^{t-1}, \mathbf{x}^t) = 0 \\ B_2^t &:= D_2 L_1(\mathbf{x}^t, \mathbf{y}^t) + D_1 L_2(\mathbf{y}^t, \mathbf{x}^{t+1}) = 0, \end{aligned}$$

where $D_{1,2}$ denote the derivatives with respect to the first and second argument, respectively.

We may consider $L_1(\mathbf{x}^t, \mathbf{y}^t)$ as a functional on covariant phase space, i.e., the space of all solutions to the equations of motion, or equivalently, the space of initial conditions on Cauchy paths. Denoting by $\mathbf{d}$ the exterior derivative on this space, we get:

$$\begin{aligned} \mathbf{d}L_1(\mathbf{x}^t, \mathbf{y}^t) &= -\mathbf{d}L_2(\mathbf{y}^{t-1}, \mathbf{x}^t) + B_1^t \mathbf{d}\mathbf{x}^t + B_2^t \mathbf{d}\mathbf{y}^{t-1} \\ &\quad - D_2 L_1(\mathbf{x}^{t-1}, \mathbf{y}^{t-1}) \mathbf{d}\mathbf{y}^{t-1} + D_2 L_1(\mathbf{x}^t, \mathbf{y}^t) \mathbf{d}\mathbf{y}^t \end{aligned} \tag{2.3.15}$$

On the space of solutions, $B_{1,2}^t$ vanish. Hence:

$$0 = \mathbf{d}^2 L_1(\mathbf{x}^t, \mathbf{y}^t) = \mathbf{d}\Big( D_2 L_1(\mathbf{x}^t, \mathbf{y}^t) \mathbf{d}\mathbf{y}^t \Big) - \mathbf{d}\Big( D_2 L_1(\mathbf{x}^{t-1}, \mathbf{y}^{t-1}) \mathbf{d}\mathbf{y}^{t-1} \Big)$$

Thus, $\omega := -\frac{1}{2}\mathbf{d}\Big( D_2 L_1(\mathbf{x}^t, \mathbf{y}^t) \mathbf{d}\mathbf{y}^t \Big)$ defines a presymplectic structure (i.e., a closed, possibly degenerate two-form on phase space) which is invariant under the evolution. This presymplectic structure will be symplectic if there is no gauge symmetry [2]. Similarly, we may define another symplectic structure using $L_2$.

If, as in the applications we have in mind, the functions $L_i$ are invariant under translations in space directions, it makes sense to compare $L_1$ and $L_2$, by identifying $\mathcal{C}_e^t$ with $\mathcal{C}_o^t$ by half-shift in light-cone direction either to the right or to the left: Due to the translation invariance, both identifications yield the same result.

For the action (2.3.11) we may choose $L_1 = L_2 =: L$ with

$$L(\mathbf{v},\mathbf{w})=\sum_{k=0}^{p-1} v_k(w_k-w_{k+1})+V(w_k-w_{k+1})+w_0(v_{p-1}-v_{-1})+\frac{1}{2}(v_{p-1}-v_{-1})(w_p-w_0)$$

and the identifications

$$v_k = g_{2t-1,2k+1}, \qquad w_k = g_{2t,2k}$$

for $L_1$, and

$$v_k = g_{2t,2k}, \qquad w_k = g_{2t+1,2k-1}$$

for $L_2$.

Thus, we get a translation invariant symplectic structure

$$\begin{aligned}
\omega=&-\frac{1}{2}\Big(\sum_{k=0}^{p-1}\mathbf{d}g_{2t-1,2k+1}\wedge(\mathbf{d}g_{2t,2k}-\mathbf{d}g_{2t,2k+2})\\
&+(\mathbf{d}g_{2t-1,2p-1}-\mathbf{d}g_{2t-1,-1})\wedge\mathbf{d}g_{2t,0}+\frac{1}{2}(\mathbf{d}g_{2t-1,2p-1}-\mathbf{d}g_{2t-1,-1})\wedge(\mathbf{d}g_{2t,2p}-\mathbf{d}g_{2t,0})\Big)\\
=&-\frac{1}{2}\Big(\sum_{k=0}^{p-2}\mathbf{d}g_{2t-1,2k+1}\wedge(\mathbf{d}g_{2t,2k}-\mathbf{d}g_{2t,2k+2})+\mathbf{d}g_{2t-1,2p-1}\wedge(\mathbf{d}g_{2t,2p-2}-\mathbf{d}g_{2t,0}-\mathbf{d}m_{2t}^{(1)})\\
&+\mathbf{d}m_{2t-1}^{(2)}\wedge\mathbf{d}g_{2t,0}+\frac{1}{2}\mathbf{d}m_{2t-1}^{(2)}\wedge\mathbf{d}m_{2t}^{(1)}\Big)
\end{aligned}$$

for $\mathcal{C}_e^t$, and similarly for $\mathcal{C}_o^t$. The corresponding Poisson structure exactly coincides with the one given in [22]:

$$\begin{array}{rcll}
\{g_{t,i},g_{t,k}\} &=& 0, & \text{if } i-k \text{ even}\\
\{g_{\tilde{t},i},g_{t,k}\} &=& 1, & \text{if } i-k \text{ odd, } i<k,\ |i-k|<2p,\\
\{g_{\tilde{t},i},m_t^{(k)}\} &=& 0 & \text{if } i-k \text{ odd,}\\
\{g_{\tilde{t},i},m_t^{(k)}\} &=& 2 & \text{if } i-k \text{ even}\\
\{m_t^{(1)},m_{\tilde{t}}^{(2)}\} &=& 0 &
\end{array} \tag{2.3.16}$$

The poisson relations between the difference or face variables is given by the definition $p_{t,k} = g_{t,k-1} - g_{t,k+1}$ and by (2.3.16):

$$\{p_{t,j},p_{t,j+1}\} = 2 \tag{2.3.17}$$

$$\{p_{t,j},p_{j+1+k}\} = 0 \quad \text{for} \quad j\in\{1\dots 2p-3\} \tag{2.3.18}$$

We note that for even period $p$ and an even potential $V(x) = V(-x)$ all these results may be easily transformed to the sign convention in the usual Hirota equation by a transformation replacing $u$ by $-u$ on each second diagonal, either on even or odd diagonals in a fixed direction. If we denote the former tranformation by $T_1$, the latter by $T_2$, space-like shifts by $\sigma$ and a global change of sign by $I$, then we have: $T_1 = I \circ T_2, \;\; T_2 = \sigma^{-1} \circ T_1 \circ \sigma$. As $L \circ I = L$ for even potential $V$, the new Lagrangian is $\tilde{L} = L \circ T_1 = L \circ T_2$, and it is again invariant under translations:

$$\tilde{L} \circ \sigma = L \circ T_1 \circ \sigma = L \circ \sigma \circ T_2 = L \circ T_2 = \tilde{L},$$

where we have used the translation invariance of the original Lagrangian $L$.

Note that the notion of quasiperiodicity is somewhat modified by this prescription: We no longer have $g_{t,2p+i} - g_{t,i} = const$ but $g_{t,2p+i} - g_{t,i} = (-1)^{[\frac{i}{2}]} const$. In this case, however, this is the natural generalization of quasiperiodicity, as this definition is compatible with the evolution equation, whereas the naive definition (not including the sign) is not.

### 2.3.2 Marsden-Weinstein reduction

The process of going over from the variables $g_{t,k}$ to the difference variables $p_{t,k} := g_{t,k-1} - g_{t,k+1}$ may be interpreted as a Marsden Weinstein reduction (see [45, 26] and previous section): If we identify the covariant phase space with the space $M$ of quasi-periodic initial conditions on a Cauchy path $\mathcal{C}^t_e$, then an action of $\mathbf{G} = \mathbb{R} \times \mathbb{R}$ on $M$ is defined by:

$$\begin{array}{lcl} (\alpha, \beta) \cdot g_{2t,2k} & := & g_{2t,2k} + \alpha \\ (\alpha, \beta) \cdot g_{2t+1,2k+1} & := & g_{2t+1,2k+1} + \beta \end{array} \tag{2.3.19}$$

for arbitrary $k \in \mathbb{Z}$. Following example 2.1.1, there exists a momentum map $\mathcal{J} : M \to \mathcal{G}^* = \mathbb{R} \oplus \mathbb{R}$ to this action,whose components are simply

$$\mathcal{J}_{(1,0)} = \frac{1}{2} m_t^{(2)}, \quad \mathcal{J}_{(0,1)} = \frac{1}{2} m_t^{(1)}.$$

This momentum map is trivially $Ad^*$-equivariant, as the group is abelian and $m_t^{(1)}, m_t^{(2)}$ Poisson-commute. Hence, for arbitrary $\rho, \mu \in \mathbb{R}$, we may consider the Marsden-Weinstein reduced phase-space

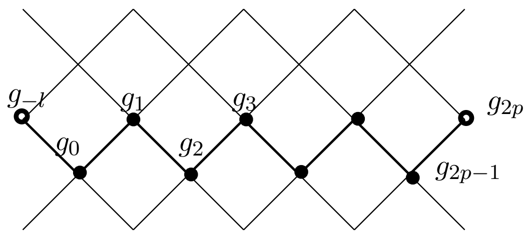
$$M_{(\rho,\mu)} = \mathcal{J}^{-1}((\rho, \mu))/(\mathbb{R} \times \mathbb{R})$$

which is a symplectic manifold again: It is just “the space of the difference variables $x$ for fixed monodromy”.

# 2.4 Dynamics of edge-Hirota type

## 2.4.1 Definition of edge algebra

We want to define functions $w_k, x_k$ on the (vertex) covariant phase space introduced in section 2.3, which was parametrized by the vertex coordinates $\{(g_{t,k})_{t\in\mathbb{Z},k\in\{-1,\ldots p-1\},k-t \text{ even}}\}$. Let us abreviate the notation of the vertex coordinates along a Cauchy path to $\{(g_k)_{t\in\mathbb{Z},k\in\{-1,\ldots 2p\}}\}$




Define

$$h_k := g_k - c_{k-1},$$

where $c_k$ shall be arbitrary functions on phase space with translational invariant monodromies $m_c^i := c_{2p+2k+1-i} - c_{2k+1-i}$.
Define

$$\begin{aligned} w_k &:= g_k - g_{k-1} + c_k - c_{k-1} & x_k &:= g_k + g_{k-1} - c_k - c_{k-1} \\ &= h_k - h_{k-1} + o_{k-1} & &= h_k + h_{k-1} - o_{k-1} \end{aligned} \tag{2.4.20}$$

for $k \in \mathbb{Z}$ where $c_k$ shall be arbitrary functions on phase space with translational invariant monodromies $m_c^i := c_{2p+2k+1-i} - c_{2k+1-i}$, $o_k = c_{k+1} - c_{k-1}$ and

$$h_k := g_k - c_{k-1}. \tag{2.4.21}$$

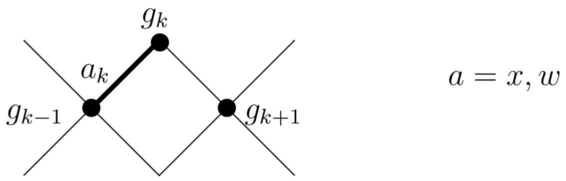
Note that $o_{k+2p} = o_k$.
The monodromies of the vertex algebra $\{(g_k)_{k\in\{-1,\ldots 2p\}}\}$ of section 2.3 were defined as $g_{2p+2k+1-i} := g_{2k+1-i} + m_g^i$ (see also 2.3.10). Hence also the variables $\{(h_k)_{k\in\{-1,\ldots 2p\}}\}$ form a (vertex) quasiperiodic phase space, where;

$$m_h^i := m_g^i - m_c^{i-1}$$

The same construction applies to the variables $\{(w_k)_{k\in\{0,\ldots 2p\}}, (x_k)_{k\in\{0,\ldots 2p\}}\}$. We get even and odd translational invariant monodromies:

$$w_{2p+2k+1-i} - w_{2k+1-i} = m_h^i - m_h^{i-1} =: (-1)^{i+1} m_w \tag{2.4.22}$$
$$x_{2p+2k+1-i} - x_{2k+1-i} = m_h^1 + m_h^2 =: m_x \tag{2.4.23}$$

with $m_w = m_h^1 - m_h^2$, and $i = 1, 2$. Since $w_k$ and $x_k$ are functions whose values depend on the values of adjacent vertex variables $g_k, g_{k-1}$, they will be from now on called **edge variables**. The algebra generated by the $4p+2$ functions $\{(w_k)_{k\in\{0,\ldots 2p\}}, (x_k)_{k\in\{0,\ldots 2p\}}\}$ (dependent on the choice of the variables $c_k$) will correspondingly be called **edge algebra**.




Note that the previously introduced face variables $p_{t,k}$ look in terms of the edgevariables as

$$p_k = g_{k-1} - g_{k+1} = -\frac{1}{2}(w_k + w_{k+1} + x_{k+1} - x_k) \tag{2.4.24}$$

Clearly the functions $\{(w_k)_{k\in\{0,...2p\}}, (x_k)_{k\in\{0,...2p\}}\}$, as defined in (2.4.20)aren't independent, in particular:

$$h_k = \frac{1}{2}(x_k + w_k) = \frac{1}{2}(x_{k+1} - w_{k+1}) + c_{k+1} - c_{k-1} \tag{2.4.25}$$

Nevertheless, since $o_{2p+k} = o_k$ and since the monodromies cancel the constraints defined by

$$\rho_k := x_k + w_k + w_{k+1} - x_{k+1} - 2c_{k+1} + 2c_{k-1} = 0 \tag{2.4.26}$$

are periodic in $k$, i.e. $\rho_{2p+k} = \rho_k$.
We understand from the above construction that if the $2p$ variables $o_k$ would be constants on phase space then the $2p+2$ dimensional covariant phase space parametrized by the $2p+2$ coordinates $\{g_k\}_{k\in\{0...2p-1\}}$ of section 2.3 could be equally be parametrized by $4p+2$ coordinates $\{(w_k)_{k\in\{0,...2p\}}, (x_k)_{k\in\{0,...2p\}}\}$ restricted to the linear subspace given by the $2p$ linear independent constraints $\rho_0 = 0, \ldots \rho_{2p-1} = 0$.

If furthermore in this case the commutation relations of $\{g_k\}_{k\in\{0...2p-1\}}$ with $c_0$ and $c_1$ (as functions in the coordinates $g_k$) are given one finds immediately the poisson relations between the new coordinates $\{(w_k)_{k\in\{0,...2p\}}, (x_k)_{k\in\{0,...2p\}}\}$.

### 2.4.2 Poisson bracket relations on the free edge algebra

In appendix (5.1) we derive a description of phase space in terms of the "free" edge algebra, i.e the algebra generated by the functions $\{(w_k)_{k\in\{0,...2p\}}, (x_k)_{k\in\{0,...2p\}}\}$ or rather $\{(u_k)_{k\in\{0,...2p\}}, (v_k)_{k\in\{0,...2p\}}\}$ where

$$v_k := \frac{1}{2}x_k \qquad u_k := \frac{1}{2}w_k \tag{2.4.27}$$

on $\mathbb{R}^{4p+2}$ which determine the periodic ultralocal face variables $\{(p_k)_{k\in\{0,...2p-1\}}\}$ via $p_k := -\frac{1}{2}(w_k + w_{k+1} - x_k + x_{k+1})$ but which are not restricted to be constant along any constraint.

The $4p+2$ variables $\{(u_k)_{k\in\{0,...2p\}},(v_k)_{k\in\{0,...2p\}}\}$ $\{(u_k)_{k\in\{0,...2p\}},(v_k)_{k\in\{0,...2p\}}\}$ or will also be called **free edge** variables. The algebra generated by $\{(u_k)_{k\in\{0,...2p\}},$ $(v_k)_{k\in\{0,...2p\}}\}$ will consequently be called **free edge algebra**.
The commutation relations of the ultralocal periodic face variables $\{(p_k)_{k\in\{0,...2p-1\}}\}$ $p_k=-(u_k+u_{k+1}-v_k+v_{k+1})$ read:

$$\begin{aligned}\{p_k,p_{k+1}\} &= 8a \\ \{p_k,p_{k+1+j}\} &= 0 \qquad \text{for} \quad j\in\{1...2p-3\} \\ p_k &= p_{2p+k}\end{aligned} \tag{2.4.28}$$

Based on certain ultralocality conditions (see 5.1.5-5.1.10) and the above pregiven poisson relations on the face algebra we derive in (5.1) a two parameter family of possible poisson relations between the generators of the free edge algebra:

$$\begin{aligned}\{u_n,u_{n+2l}\} &= -2a-b+c \qquad l\in\{1...p-1\} \\ \{u_n,u_{n+2m-1}\} &= 2a+b-c \qquad m\in\{1...p\} \\ \{v_n,v_{n+k}\} &= 2a-b-c \qquad k\in\{1...2p-1\} \\ \{u_n,v_{n+j}\} &= 0 \qquad j\in\mathbb{Z}-\{0\} \mod 2\pi \\ \{u_n,v_n\} &= -2a-c \qquad n\in\mathbb{Z} \\ \{u_n,m_u\} &= (-1)^{n+1}(4a+2b-2c) \\ \{u_n,m_v\} &= 0 \\ \{v_n,m_u\} &= 0 \\ \{v_n,m_v\} &= 4a-2b-2c.\end{aligned} \tag{2.4.29}$$

The commutation relation with the functions $o_k=u_k+u_{k+1}+v_k-v_{k+1}$ are:

$$\begin{aligned}\{u_n,o_n\} &= b-2c \\ \{u_{n+1},o_n\} &= -b+2c \\ \{v_n,o_n\} &= b+2c \\ \{v_{n+1},o_n\} &= b+2c \\ \{a_{n+k},o_n\} &= 0 \quad \text{f. a.} \quad k\in\{2\dots 2p-1\}\end{aligned} \tag{2.4.30}$$

We see immediately that we obtain the standard poisson bracket on $\mathbb{R}^{4p+2}/\mathbb{Z}^{4p+2}$,i.e.

$$\{v_k,u_j\}=4a\delta_{kj} \quad mod\, 2p \quad \{v_k,v_j\}=0 \quad \{u_j,u_k\}=0. \tag{2.4.31}$$

if we set $b=0,c=2a$.
We also see immediately that the constraint functions $o_k$ lie only in the center of the edge algebra iff $b=0,c=0$ in this case we precisely induce the poisson relations (2.3.16) between the vertex variables $\{(g_k)_{k\in\{-1,...2p\}}\}$ via definition (2.4.25) if $c_k=const$,i.e

$$g_k := \frac{1}{2}(x_k+w_k)+c_{k-1} \qquad c_k=const. \tag{2.4.32}$$

### 2.4.3 Ultralocal poisson structure and gauge action

By definition 2.1.3 any function $\rho$ on phase space $\mathcal{M}$ gives rise to a hamiltonian vector field $\xi_\rho$ on $\mathcal{M}$:

$$(\xi_\rho)^l = \sum_m \eta^{lm} \frac{\partial \rho}{\partial q^m},$$

where $\eta$ is the two tensor which defines the poisson bracket on $\mathcal{M}$. $\xi_\rho$ is the generator of an infinitesimal transformation $\phi_\rho : (-\epsilon, \epsilon) : \mathcal{M} \to \mathcal{M}$ :

$$\phi_\rho(\alpha, p) := exp(\alpha \xi_\rho)|_p,$$

where $exp$ denotes here the usual exponential map $exp : U \subset T_p\mathcal{M} \to V \subset \mathcal{M}$. Now let $F \in C^\infty(\mathcal{M})$ be a smooth function on $\mathcal{M}$, we find

$$\frac{d}{d\alpha}|_0 F(\phi_\rho(\alpha, p)) = \sum_i \frac{\partial F}{\partial q_i} dq_i(\xi_\rho)|_p = \{F, \rho\}|_p.$$

In our example where $\mathcal{M} = \mathbb{R}^{4p+2}$ and the poisson bracket is the above **ultralocal** poisson bracket (2.4.31) the flow induced by the hamiltonian vectorfields $\xi_{\rho_k}$

$$dq^l(\xi_{o_k}) = 4a \sum_{m=0}^{4p+1} J^{lm} \frac{\partial o_k}{\partial q^m}$$

where

$$\begin{aligned} q_i &:= u_i \quad \text{for} \quad i \in 0, 2 \ldots 2p \\ q_i &:= v_{i-(2p+1)} \quad \text{for} \quad i \in 2p+1 \ldots 4p+1 \end{aligned}$$

and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \text{with} \quad I = \begin{pmatrix} 1 & 0 & \ldots & 0 & 1 \\ 0 & 1 & & & 0 \\ . & 0 & . & & 0 \\ 1 & & & 0 & 1 \end{pmatrix} \tag{2.4.33}$$

is easily found by (for simplicity $a = -\frac{1}{4}$); $o_k = u_k + u_{k+1} + v_k - v_{k+1}$

$$\begin{aligned} \frac{d}{d\alpha_k} u_{n+2pj}(\phi_{o_k}(\alpha_k, p)) &= \{u_{n+2pj}, o_k\}|_p = \delta_{n,k} - \delta_{n-1,k} = const. \in \mathbb{R} \\ \frac{d}{d\alpha_k} v_{n+2pj}(\phi_{o_k}(\alpha_k, p)) &= \{v_{n+2pj}, o_k\}|_p = -\delta_{n,k} - \delta_{n-1,k} = const. \in \mathbb{R} \\ & \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z} \; mod \, 2p \quad n \in \{0 \ldots 2p-1\} \end{aligned}$$

Define

$$u_{n+2pj}(\alpha)(p) := u_{n+2pj}(p) + \alpha_n - \alpha_{n-1} \tag{2.4.34}$$
$$v_{n+2pj}(\alpha)(p) := v_{n+2pj}(p) - \alpha_n - \alpha_{n-1}. \tag{2.4.35}$$

Hence we found that the constraint functions $o_k$ induce an action of the commutative group $\mathcal{G} = \mathbb{R}^{2p}$ on phase space $\mathbb{R}^{4p+2}$ (supplied with the above ultralocal poisson structure):

$$\begin{aligned} \phi \ : \ & \mathcal{G} \times \mathcal{M} \to \mathcal{M} \\ & (g = (\alpha_0, \dots, \alpha_{2p-1}), p = \textstyle\sum_{j=0}^{4p+1} p_j e_j) \mapsto \\ & \phi(g,p) = \textstyle\sum_{j=0}^{2p} u_j(\alpha)(p) e_j + \sum_{j=2p+1}^{4p+1} v_{j-2p+1}(\alpha)(p) e_j \end{aligned}$$

where $\alpha_{2p+k} := \alpha_k$.
The $2p$ infinitesimal generators of the action $\phi$, which are

$$\begin{aligned} \xi_{o_k} &= \frac{\partial}{\partial \alpha_k} \phi((\alpha_0, \dots, \alpha_{2p-1}), p) = e_k - e_{k+1} - e_{k+1+2p} - e_{k+2+2p} \\ \xi_{o_0} &= e_0 - e_1 + e_{2p} - e_{2p+1} - e_{2p+2} - e_{4p+2} \qquad k \in \{1, \dots, 2p\ -\ 1\} \end{aligned}$$

are the hamiltonian vectorfields of the functions $o_k$, $k \in \{0, \dots, 2p-1\}$, hence we can define a momentum mapping $J : \mathcal{M} \to \mathfrak{g}^*$ by

$$J := \sum_{j=0}^{2p-1} o_j d\alpha_j.$$

J is not equivariant since $Ad^*_{g^{-1}} J(\xi) = J(\xi)$ but

$$J(\xi_l)(\phi_g(p)) = J(\xi_l)(p) + \alpha_{l+1} - \alpha_{l-1}.$$

The map $s : \mathcal{G} \to \mathfrak{g}^*$:

$$(\alpha_1, \dots, \alpha_{2p-1}) \mapsto \sum_{j=0}^{2p-1} (\alpha_{j+1} - \alpha_{j-1}) d\alpha_j$$

is called a **coadjoint cocycle** associated to $J$.

Since $\mathcal{G}$ is commutative the appearance of the above cocycle results from the fact that the constraint functions $o_k$ do not commute with each other and hence the constraint function $o_j$ is not invariant under the flow which is induced by the hamiltonian vectorfield $\xi_{o_k}$, $k \neq j$.

Let us search for other functions on phase space which are possibly invariant under the flow of $\phi$. We find that the $2p$ dimensional linear subspace spanned by the hamiltonian vector fields $\xi_{o_k}$ is given by the following $2p+2$ level sets:

$$p_k := -(u_k + u_{k+1} - v_k + v_{k+1}) = const \tag{2.4.36}$$

and

$$\sum_{0}^{p-1} p_{2k} + m_c^1 = m_u - m_v = const \qquad \sum_{0}^{p-1} p_{2k+1} + m_c^2 = -(m_u + m_v) = const. \tag{2.4.37}$$

In other words the functions $(p_k)_{k\in\{0\ldots 2p-1\}}$, $m_u - m_v$ and $m_u + m_v$ are invariant under the flow $\phi$ and their gradients are linearily independent.

The monodromies $m_u$ and $m_v$ of (2.4.23) are trivially invariant under the flow of $\phi$ since they poisson commute with all coordinates $\{(u_k)_{k\in\{0,\ldots 2p\}}, (v_k)_{k\in\{0,\ldots 2p\}}\}$, (2.4.31) i.e. they are in the null space of $J$ (2.4.33). Let us set them without losing generality to be equal to zero

$$m_a^i = 0.$$

(The argumentation works analogously if we introduce another constant monodromy). We reduce our phase space to the $4p$ dimensional phase space parametrized by the periodic coordinates:

$$\begin{aligned} u_{i+2p} &= u_i \\ v_{i+2p} &= v_i \end{aligned}$$

Now the poisson structure isn't anymore degenerate, since we eliminated the null space of $J$, i.e. we restrict ourselves to a $4p$ dimensional torus.
Let us consider the L-matrices of chapter 1.3:

$$L_n(\lambda \pm \omega) = \begin{pmatrix} e^{iu_n} & -e^{\lambda\pm\omega} e^{-iv_n} \\ e^{\lambda\pm\omega} e^{iv_n} & e^{-iu_n} \end{pmatrix}$$

If the entries $u_n, v_n$ in $L_n(\lambda \pm \omega)$ are identified with our previously introduced edge variables on $4p$ dimensional phase space then we can lift the action $\phi$:

$$\begin{aligned} \mathbb{R}^{2p} \times \mathcal{M} &\rightarrow \mathcal{M} \\ (\alpha = (\alpha_0 \ldots \alpha_{2p-1}), q = \sum_{j=0}^{2p-1} q_j e_j^u + \sum_{j=0}^{2p-1} q_{j+2p} e_j^v) &\mapsto q(\alpha) \\ &= \sum_{j=0}^{2p-1} (q_j + \alpha_j - \alpha_{j-1}) e_j^u \\ &\quad + \sum_{j=0}^{2p-1} (q_{j+2p} - \alpha_j - \alpha_{j-1}) e_j^v \end{aligned}$$

to an gauge action on the L-matrices, namely

$$\begin{aligned} L_n^\alpha(\lambda \pm \omega) &= \begin{pmatrix} e^{iu_n(\alpha)} & -e^{\lambda\pm\omega} e^{-iv_n(\alpha)} \\ e^{\lambda\pm\omega} e^{iv_n(\alpha)} & e^{-iu_n(\alpha)} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} e^{i\alpha_n} & 0 \\ 0 & e^{-\alpha_n} \end{pmatrix}}_{A_{n-1}(\alpha)} \begin{pmatrix} e^{iu_n} & -e^{\lambda\pm\omega} e^{-iv_n} \\ e^{\lambda\pm\omega} e^{iv_n} & e^{-iu_n} \end{pmatrix} \begin{pmatrix} e^{-i\alpha_{n-1}} & 0 \\ 0 & e^{\alpha_{n-1}} \end{pmatrix} \\ &= A_n(\alpha) L_n(\lambda \pm \omega) A_{n-1}(\alpha)^{-1} \end{aligned}$$

Note that the matrix

$$M := A_{2p}L_{2p}L_{2p-1}\dots L_1 A_0 \tag{2.4.38}$$

carries the gauge information only in its off diagonals, since $\alpha_{2p} = \alpha_0$. $M$ will be called monodromy matrix. The meaning of the face variables $p_k$ and the variables $o_k$ (which as will be shown later also "live") on faces of the space-time lattice) becomes now apparent. The variables $p_k$ (as functions in the (free) edge variables) are invariant under any gauge transformation in the space of L-matrices. They play the role of what would in physics be called observables. The variables $o_k$ on the other hand exactly reflect the relative gauge freedom which is left after fixing "initial" gauges $\alpha_0$ and $\alpha_1$:

$$o_k(\alpha) = o_k(0) + 2\alpha_{k+1} - 2\alpha_{k-1}$$

Moreover in the above ultralocal poisson structure the functions $o_k$ generate all gauge transformations in the space of L-matrices.

Since the diagonals of the monodromy matrix are gauge invariant, they will be expressable by the $2p+2$ gauge invariant variables $(p_i)_{i\in\{0\dots 2p\}}$ and $m_u \pm m_v$ (cf. (2.4.36,2.4.37)).

If we switch now to a poisson structure, where the variables $o_k$ are constant (i.e. the poisson structure given by (2.4.29) for $b=0, c=0$) then this means that we fix the relative gauges. The variables $c_k$ are not fully representable in terms of the edge algebra, since $c_0$ and $c_1$ are a priori undeterminable.

$$\begin{aligned} c_{2k} &= \sum_{j=1}^{2k} u_j - \sum_{j=1}^{2k} (-1)^j v_j + c_0 \\ c_{2k+1} &= \sum_{j=2}^{2k+1} u_j + \sum_{j=2}^{2k+1} (-1)^j v_j + c_1 \end{aligned}$$

$c_0$ and $c_1$ shall be viewed as carrying the rest of the gauge freedom. In particular they might be even not expressable in terms of functions on phase space. That means if we choose $c_0$ and $c_1$ to be constant then we restrict ourselves to a special class of gauge fixings.

In the following section we would like to investigate this point a little further.

### 2.4.4 Introducing a larger phase space

If we fix $4p+2$ edge variables $\{(u_k)_{k\in\{0,\dots 2p\}}, (v_k)_{k\in\{0,\dots 2p\}}\}$ along a Cauchy path then this determines a set of initial L-matrices. The zerocurvature condition (1.3.27)doesn't define an evolution for the edge variables $u_k, v_k$ because of the gauge freedom. So the $u_k$ and $v_k$ are in this sense no coordinates on a phase space of solutions, but rather coordinates on a "pseudo" phase space, which has yet to be reduced to physical phase space.

But let us first extend the above pseudo phase space endowed with the coordinates $\{(u_k)_{k\in\{0,\dots 2p\}}, (v_k)_{k\in\{0,\dots 2p\}}\}$. We imagine the coordinates $u_k, v_k$ as being difference variables between $4p+4$ vertex variables $\{(l^u_k)_{k\in\{-1,\dots 2p\}}, (l^v_k)_{k\in\{-1,\dots 2p\}}\}$. which live on an extend space $\mathcal{PM}$ i.e.:

$$\begin{aligned} u_k &:= l^u_k - l^u_{k-1} \\ v_k &:= l^v_k + l^v_{k-1} \end{aligned}$$

We define an action $\phi$ of the group $\mathcal{G} = \mathbb{R}^{4p+4}$ on $\mathcal{PM}$ by the following: Let $p \in \mathcal{PM}$, $p = \sum_{i=-1}^{2p} l^u_k e^u_k + \sum_{i=-1}^{2p} l^v_k e^v_k$, where $e^u_k, e^v_k$ form the standard orthonormal basis on $\mathbb{R}^{4p+4}$ then

$$\phi((\alpha_{-1}\dots\alpha_{2p}, \beta_{-1}\dots\beta_{2p}), p) = \sum_{i=-1}^{2p}(l^u_k + \alpha_k)e^u_k + \sum_{i=-1}^{2p}(l^v_k - \beta_k)e^v_k$$

The action of the subgroup $\mathcal{G}_C \subset \mathcal{G}$, where

$$\mathcal{G}_C := (\alpha\dots\alpha, 0\dots 0) \cup (0\dots 0, \beta, -\beta, \beta, -\beta\dots\beta)$$

leaves the functions $u_k, v_k$ invariant. The action of $\mathcal{G}_C$ on $\mathcal{PM}$ is a consequence of the integration process, as will get clear in (2.4.39).

The action of the subgroup $\mathcal{G}_G \subset \mathcal{G}$ defined by

$$\mathcal{G}_G = (\alpha_{-1}\dots\alpha_{2p}, \alpha_{-1}\dots\alpha_{2p})$$

induces the gauge actions (2.4.34) in the space of L-matrices.
Clearly $\mathcal{G}_C \cap \mathcal{G}_G = \{0\}$.
Let us define

$$\begin{aligned} g_k &:= l^u_k + l^v_k \quad \Rightarrow \quad l^u_k = \tfrac{1}{2}(g_k + c_k) \\ c_k &:= l^u_k - l^v_k \qquad\quad\;\; l^v_k = \tfrac{1}{2}(g_k - c_k). \end{aligned}$$

The functions $g_k$ (as functions in the coordinates on $\mathcal{PM}$) are invariant under the action of $\mathcal{G}_G$, whereas:

$$c_k(\phi((\alpha_{-1}\dots\alpha_k\dots\alpha_{2p}, \alpha_{-1}\dots\alpha_k\dots\alpha_{2p}), p)) = c_k(p) + 2\alpha_k$$

The functions $g_k$ and $c_k$ transform under an action of $\mathcal{G}_C$ as:

$$\begin{aligned} g_k(\phi((\alpha\dots\alpha, \beta, -\beta\dots\beta), p)) &= g_k(p) + (\alpha + (-1)^k\beta) \\ c_k(\phi((\alpha\dots\alpha, \beta, -\beta\dots\beta), p)) &= c_k(p) + (\alpha - (-1)^k\beta) \end{aligned}$$

We find that the action (2.4.39) corresponds to the action induced by the monodromies in (2.3.19).
By the above it becomes clear that a gauge fixing in the space of L-matrices is defined by $2p+2$ independent constraints

$$f_j(\{g_k\}) - c_j = 0.$$

So we find that the gauge fixing in [7] corresponds to the case

$$c_k = g_k - \frac{\pi}{2}$$

whereas in [30] to the case $c_{k-1} = g_k$. In the turn we had already considered a fixing, which is given by

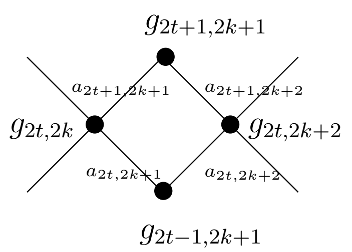
$$c_k = const.$$

.

### 2.4.5 Evolution within the edge algebra

Let us investigate the dynamical structure of our just introduced system. This means, we investigate the consequences of the definitions:
Let $t, k \in \mathbb{Z}$,

$$\begin{array}{ll} \text{for} \quad k-t \quad \text{even}: & \\ w_{t,k} = g_{t,k} - g_{t-1,k-1} + c_{t,k} - c_{t-1,k-1} & x_{t,k} = g_{t,k} + g_{t-1,k-1} - c_{t,k} - c_{t-1,k-1} \\ \text{for} \quad k-t \quad \text{odd}: & \\ w_{t,k} = g_{t-1,k} - g_{t,k-1} + c_{t-1,k} - c_{t,k-1} & x_{t,k} = g_{t-1,k} + g_{t,k-1} - c_{t-1,k} - c_{t,k-1} \end{array} \tag{2.4.39}$$




$a = x, w$

together with the evolution for the gauge invariant part of the above pseudo phase space:

$$g_{t+2,k} - g_{t,k} = V'(g_{t+1,k-1} - g_{t+1,k+1})$$

which was already defined in (2.3.8).
Definition (2.4.39) gives immediately (for $k-t$ even, $t, k \in \mathbb{Z}$):

$$g_{t,k} = \frac{1}{2}(x_{t,k} + w_{t,k}) + c_{t-1,k-1} \tag{2.4.40}$$

$$= \frac{1}{2}(x_{t+1,k+1} - w_{t+1,k+1}) + c_{t+1,k+1} \tag{2.4.41}$$

$$g_{t,k} = \frac{1}{2}(x_{t+1,k} + w_{t+1,k}) + c_{t+1,k-1} \tag{2.4.42}$$

$$= \frac{1}{2}(x_{t,k+1} - w_{t,k+1}) + c_{t-1,k+1} \tag{2.4.43}$$

From where we can read off the constraints

$$\rho_{t,k} := x_{t,k} + w_{t,k} - x_{t,k+1} + w_{t,k+1} - 2o_{t-1,k} = 0 \qquad t,k \in \mathbb{Z} \quad k-t \quad \text{even}$$

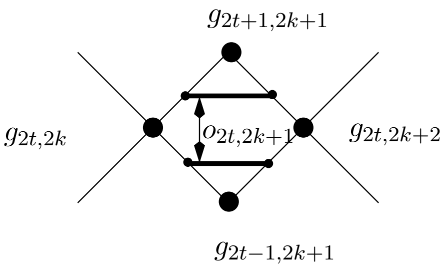
(where $o_{t-1,k} := c_{t-1,k+1} - c_{t-1,k-1}$, $k-t$ even) which were introduced in the previous section as constraints along the Cauchypath $\mathcal{C}_t$. Equations (2.4.41) and (2.4.43) have to be interpreted as additional evolution equations since they connect coordinates along Cauchy paths at different times.
From (2.4.41) and (2.4.43) we obtain immediately (for $k-t$ even):

$$\begin{aligned} p_{t+1,k} &= g_{t+1,k-1} - g_{t+1,k+1} && (2.4.44)\\ &= \frac{1}{2}(x_{t+2,k} - x_{t+2,k+1} - w_{t+2,k} - w_{t+2,k+1}) && (2.4.45)\\ &= \frac{1}{2}(x_{t+1,k} - x_{t+1,k+1} - w_{t+1,k} - w_{t+1,k+1}) && (2.4.46) \end{aligned}$$

and

$$\begin{aligned} o_{t+1,k} &= c_{t+1,k+1} - c_{t+1,k-1} && (2.4.47)\\ &= \frac{1}{2}(x_{t+2,k} - x_{t+2,k+1} + w_{t+2,k} + w_{t+2,k+1}) && (2.4.48)\\ &= \frac{1}{2}(x_{t+1,k} - x_{t+1,k+1} + w_{t+1,k} + w_{t+1,k+1}) && (2.4.49) \end{aligned}$$

which shows that not only the previously defined variables $p_{t,k}$ but also the variables $o_{t,k}$ "live" on the faces of our space time, which in the turn justifies their definition as **conjugate face variables**:




Equations (2.4.46,2.4.49) give finally the following evolution equations for $k-t$ even:

$$\begin{aligned} x_{t+1,k} - x_{t+1,k+1} &= x_{t+2,k} - x_{t+2,k+1} && (2.4.50)\\ w_{t+1,k} + w_{t+1,k+1} &= w_{t+2,k} + w_{t+2,k+1} && (2.4.51) \end{aligned}$$

In addition we have an evolution equation for the vertex variables, which we still have to reformulate in terms of the edge variables $\{(w_k)_{k\in\{0,\dots 2p\}}, (x_k)_{k\in\{0,\dots 2p\}}\}$ :

$$g_{t+2,k} - g_{t,k} = V'(g_{t+1,k-1} - g_{t+1,k+1}) \qquad (2.4.52)$$

For that purpose let us consider the following identities:

$$\begin{aligned} g_{t+2,k} - g_{t,k} &= \frac{1}{2}(x_{t+2,k} + w_{t+2,k}) - \frac{1}{2}(x_{t+1,k} + w_{t+1,k}) \\ &\overset{(2.4.41)}{=} \frac{1}{2}(2x_{t+2,k} - x_{t+1,k-1} - w_{t+1,k-1} - x_{t+1,k} - w_{t+1,k}) + c_{t+2,k} - c_{t,k-2} \\ &\overset{\rho_{t,k}=0}{=} x_{t+2,k} - x_{t+1,k} + c_{t+2,k} - c_{t,k} \end{aligned}$$

Analogously we collect by using (2.4.50,2.4.51) the following identities:

$$\begin{aligned} g_{t+2,k} - g_{t,k} &= x_{t+2,k+1} - x_{t+1,k+1} + c_{t+2,k} - c_{t,k} \\ &= w_{t+2,k} - w_{t+1,k} + c_{t,k} - c_{t+2,k} \\ &= -w_{t+2,k+1} + w_{t+1,k+1} + c_{t,k} - c_{t+2,k} \end{aligned} \tag{2.4.53}$$

furthermore

$$\begin{aligned} p_{t+1,k} &= g_{t+1,k-1} - g_{t+1,k+1} && (2.4.54) \\ &= x_{t+1,k} - x_{t+1,k+1} + c_{t+1,k-1} - c_{t+1,k+1} \\ &= -w_{t+1,k} - w_{t+1,k+1} + c_{t+1,k+1} - c_{t+1,k-1} && (2.4.55) \end{aligned}$$

Inserting the above identities now into (2.4.52) we obtain following system of evolution equations ($k - t$ even):

$$\begin{aligned} x_{t+2,k} &= V'(x_{t+1,k} - x_{t+1,k+1} - o_{t+1,k}) + x_{t+1,k} + c_{t,k} - c_{t+2,k} && (2.4.56) \\ x_{t+2,k+1} &= V'(x_{t+1,k} - x_{t+1,k+1} - o_{t+1,k}) + x_{t+1,k+1} + c_{t,k} - c_{t+2,k} \\ w_{t+2,k} &= V'(-w_{t+1,k} - w_{t+1,k+1} + o_{t+1,k}) + w_{t+1,k} - (c_{t,k} - c_{t+2,k}) \\ w_{t+2,k+1} &= -V'(-w_{t+1,k} - w_{t+1,k+1} + o_{t+1,k}) + w_{t+1,k+1} + (c_{t,k} - c_{t+2,k}) \end{aligned}$$

We see immediately that we can read off an evolution in the edge variables if we express the functions $c_{t,k}$ in terms of the variables $g_{t,k}$.

We will call this system of equations for obvious reasons a system of **edge-Hirota type**.

We notice that in the case $c_k = const.$ any solution $x$ (or $w$) to system (2.4.56),(2.4.57) (or system (2.4.57),(2.4.57) respectively) for a fixed choice of $2p$ constants $o_k$ determines because of (2.4.55) a solution to the evolution equation in the difference or face variables:

$$p_{t+1,k} - p_{t-1,k} = V'(p_{t,k-1}) - V'(p_{t,k+1}).$$

Remark that the time difference of two vertex variables can be expressed in terms of the edge variables for two different Cauchypaths, i.e.

$$\begin{aligned} &-\frac{1}{2}(x_{t+1,k-1} + x_{t+1,k} - w_{t+1,k-1} + w_{t+1,k}) + \frac{1}{2}(x_{t,k-1} + x_{t,k} - w_{t,k-1} + w_{t,k}) \\ &= 2(c_{t+1,k-1} - c_{t-1,k-1}) \end{aligned} \tag{2.4.57}$$

# Chapter 3

# Quantized Dynamics

We will go on a brief excursion in order to indicate what we mean by the term "quantization" and in succession suggest a quantization of the previously introduced classical models.

## 3.1 Quantization

The idea of quantization is, roughly speaking, to substitute commutative algebras such as functions on manifolds by noncommutative algebras. Canonically these noncommutative algebras are provided with some extra structure in order to establish an isomorphism between them and operators on some Hilbertspace, (like e.g. $L^2(\mathbb{R})$). The before mentioned operators on a Hilbertspace usually serve then for replacing the result of a classical measurement (value of the field variable) by a whole set of possible results of a measurement (eigenvalues of the usually self-adjoint operator). The scalar product on that Hilbert space finally is needed for the probabilistic interpretation of the model. Additional features like e.g. relativistic invariance of a theory are usually implemented in analogy with the corresponding classical theory.

There is still a lot of uncertainty about how much structure one should carry over from classical theories to their quantum mechanical counterpart. Most quantum theories try to inherit as little classical features as possible, which in the turn unfortunately broadens the availability of quantum theories.

We do not want to attempt to describe what kind of quantum theories exist and even not attempt to describe what kind of features should be provided by a "good" quantum theory. Nevertheless there are some concepts which proved to be working well especially in linear theories.

In the sequel we would like to make use of some of these well-working concepts. In this way the below "theory of quantization" will be very much adapted to the considered models and will be on discrete space-time. The axioms which will be soon set are not intended to be complete. They shall serve as a formal guidance

through the main ingrediences of the described models and shall not necessarily be viewed as a conceptional outline.
Let us carry together some basic definitions.

**Definition 3.1.1** *An algebra $A$ over a field $K$ is a vectorspace $V$ over a field $K$ endowed with a bilinear operation called multiplication:*

$$\cdot : A \times A \to A$$

**Definition 3.1.2** *[15]*

*A Banach algebra $A$ is an algebra over $\mathbb{C}$ with identity $\mathbb{1}$ which has a norm making it into a Banach space and satisfying $\|\mathbb{1}\| = 1$ and the inequality $\|fg\| \leq \|f\|\|g\|$ for $f$ and $g$ in $A$.*

**Definition 3.1.3** *[15]*

*If $A$ is a Banach algebra, then an involution on $A$ is a mapping $T \to T^*$ which satisfies:*

1.) $T^{**} = T \quad for \quad T \in A$

2.) $(\alpha S + \beta T)^* = \bar{\alpha} S^* + \bar{\beta} T^* \quad for \quad S, T \in A \quad and \quad \alpha, \beta \in \mathbb{C}$

3.) $(ST)^* = T^* S^* \quad for \quad S, T \in A.$

*If, in addition, 4.) $\|T^*T\| = \|T\|^2$ for $T \in A$, then $A$ is called a $C^*$-algebra.*

**Definition 3.1.4** *Let $M_Q = \{Q_0, ...Q_{n-1}\}$ be a finite set of unitary operators $Q_i \in \mathcal{U}(\mathcal{H})$ on a Hilbertspace $\mathcal{H}$.*
*Define $Q_i^k := Q_i \circ Q_i ... \circ Q_i$ and $Q_i^0 = \mathbb{1}$, where $\mathbb{1}$ is the identity operator on $\mathcal{H}$ and the multiplication is understood as the composition of operators in $\mathcal{L}(\mathcal{H})$ (linear operators on $\mathcal{H}$). The polynomials $P(N, Q)$ generated by finite words of $Q_i \in \mathcal{U}(\mathcal{H})$ form an algebra over $\mathbb{C}$ with identity which we denote with*

$$\mathcal{P}(Q) = \{P(N, Q) = \sum_{m=0}^{N} a_m Q_0^{m_0} Q_1^{m_1} ... Q_{n-1}^{m_{n-1}}\} \qquad a_m \in \mathbb{C},$$

$$m = (m_0 ... m_{n-1}) \in \mathbb{N}^n \cup \{0\} \qquad N = (N_0 ... N_{n-1}) \in \mathbb{N}^n \cup \{0\}$$

**Lemma 3.1.5** *The closure $\overline{\mathcal{P}(Q)}$ of $\mathcal{P}(Q)$ in the usual operator norm on $\mathcal{L}(\mathcal{H})$ is a $C^*$-algebra.*

**Proof:**
$\overline{\mathcal{P}(Q)}$ is clearly a Banachalgebra. Since $\overline{\mathcal{P}(Q)}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ relations (1.-4.) of (3.1.3) also hold for $\overline{\mathcal{P}(Q)}$. □

Let $m \in C^\infty(M)$ and $\{f_i\}$ being a set of local coordinate components on a neighbourhood $U$.

**Definition 3.1.6** *A set $G$ of real valued $C^\infty$-functions on a neighbourhood $U$ in phase space $M$ is called* **number commuting** *iff*

$$\{g_i, g_k\} = c_{ik} = const \in \mathbb{R} \qquad f.\ a. \qquad g_i \in G.$$

The following definition shall describe the quantization of a phase space $\mathcal{M}$ belonging to a discrete evolution. (confer section 2.2). The set of real number commuting $C^\infty$ functions $M_f^{\{,\}} = \{f_0...f_{n-1}\}$ appearing in the below definition shall be viewed as components of a coordinate map assigned to a neighbourhood $U \in \mathcal{M}$. In particular this means that our quantization will be coordinate dependent.

**Definition 3.1.7** *A* **quantization** $Q^h$ *on a neighbourhood $U$ on a manifold $\mathcal{M}$ is a parameter dependend $\mathbb{C}$-linear map from $C^\infty$-functions on $\mathcal{M}$ into the linear operators on some Hilbert space $\mathcal{H}$, i.e. $Q^h : C^\infty(M) \mapsto \mathcal{L}(\mathcal{H})$ subject to the following conditions, which shall hold for all parameter values of $h \in \mathbb{C}$ and for a fixed set of real numbercommuting $C^\infty$-functions $M_f^{\{,\}} = \{f_0...f_{n-1}\}$ :*
*1.* **reality condition:**

$$Q^h(g^*) = Q^h(g)^* \qquad g \in C^\infty(M)$$

*2.* **unitarity condition:** *If $s = s^*$, $\quad s \in C^\infty(M)$ then*

$$Q^h(e^{-is}) = Q^h(e^{is})^{-1}$$

*3.* **quantum condition:**

$$Q^h(e^{if_k})Q^h(e^{if_j}) = e^{ihc_{kj}}Q^h(e^{if_j})Q^h(e^{if_k}) \qquad f_i \in M_f^{\{,\}}; \quad \{f_k, f_j\} = c_{kj}$$

*4.* **factorizability condition**
*There exists an $a \in \mathbb{R}$ such that :*

$$Q^h(e^{if_k+if_j}) = e^{iha}Q^h(e^{if_k})Q^h(e^{if_j}) \qquad f_i \in M_f^{\{,\}}.$$

*5.* **completeness condition**
*Define $Q_j^h := Q^h(e^{if_j})$. Because of condition 1. and 2. the $Q_j$ are unitary operators. We will take them from now on as the generators of a $C^*$-algebra $\overline{\mathcal{P}(Q)}$ which shall be defined analogously to (3.1.4).*
*For any $g \in C^\infty(\mathcal{M})$ , $Q^h(g) \in \overline{\mathcal{P}(Q)}$*

*Remark:*
For defining a quantization, which would give a senseful semiclassical limit for the algebra of (real) observables one would certainly have to include more axioms into the definition of a quantization or even better find an explicit construction scheme. We remark that if we assume that there exists a quantization such that

$Q^h(e^{it_j f_k}) = Q^h(e^{if_k})^{t_j} =: e^{it_j q_k}$; $q_k = q_k^* \in \mathcal{L}(\mathcal{H})$, where $Q^h(e^{if_k})^{t_j}$ analytic in $t_j$ and $f_k \in M_f^{\{,\}}$ we get by extending the quantum condition (3.) of definition 3.1.7 to noninteger exponents

$$Q^h(e^{if_k})^{t_1} Q^h(e^{if_j})^{t_2} = e^{ih(t_1 t_2) c_{kj}} Q^h(e^{if_j})^{t_2} Q^h(e^{if_k})^{t_1}$$

immediately that

$$[q_k, q_j] = ih\{f_j, f_k\}\mathbb{1} \tag{3.1.1}$$

which is may be a more familiar expression in the theory of quantization (cf. e.g [56]). Nevertheless starting out with condition (3.1.1) could in principle lead into troubles (cf. Reed-Simon I, 1980, p.275). Since in the following all further investigations can be carried out by axioms 1.-5.) of (3.1.7) without introducing additional structure we will not try to define more as necessary.

Let us extend the notion of quantization to the class of $C^\infty$-functions with values in some higher dimensional vector space. Note that any $C^\infty$-tensorfield over a sufficiently small neighbourhood on a manifold expressed in local coordinates belongs to this class, i.e. this will give us the opportunity to "quantize" - at least locally - differential objects of quite general type.

**Definition 3.1.8** *A* **quantization** $\mathcal{Q}^h$ *of a* $C^\infty$*-map* $L$ *from a neighbourhood* $U$ *on a manifold* $\mathcal{M}$ *into the set of multilinear maps* $\mathcal{L}^{r+s} := L^{r+s}(V^* \dots V^*, V \dots V, \mathbb{R})$ *(r copies of* $V^*$*, s copies of* $V$*) is given by the quantization of its components, i.e.*

$$\mathcal{L}^{r+s} \times C^\infty \to \mathcal{L}^{r+s} \times \overline{\mathcal{P}(Q)}$$

$$Q^h(L)(de_{j_1} \dots de_{j_r}, e_{j_1} \dots e_{j_s}) := Q^h(L^{j_1, \dots j_r}_{j_1 \dots j_s}) \tag{3.1.2}$$

(3.1.2) is well-defined since $Q^h$ is $\mathbb{C}$-linear. We denote the set of such quantized multilinear maps with $\overline{\mathcal{P}(Q)}^{(r,s)}$. $\overline{\mathcal{P}(Q)}^{(r,s)}$ forms an algebra if we choose the multiplication map to be the compositions of linear operators on $\mathcal{L}(H)$ and $V$ or $V^*$, respectively.

## 3.2 Quantization of phase space and space time

We sofar introduced the concept of quantization without any relation to discrete space time.

Let us assume that we are given a poisson structure on a manifold which can be understood as a phase space in the sense of the previous chapters (please cf. section 2.2). This enables us to quantize local coordinate components $\{q_i^{\mathcal{C}_k}\}$ assigned to the Cauchy path $\mathcal{C}_k$ by means of the quantization procedure given in (3.1.7).

As a result we have now constructed a map $Q^h_{\mathcal{C}_k} : C^\infty(\mathcal{M}) \to \overline{\mathcal{P}(Q_{\mathcal{C}_k})}$. $Q^h_{\mathcal{C}_k}$ depends on the choice of the local coordinate components $\{q_i^{\mathcal{C}_k}\}$ assigned to the

Cauchy path $\mathcal{C}_k$ and so does the corresponding $C^*$-algebra $\overline{\mathcal{P}(Q_{\mathcal{C}_k})}$. We are left to incorporate the notion of evolution or change of coordinates on phase space within the quantum setting.

**Definition 3.2.1** *An* **evolution automorphism** $\phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}$ *(evolving from Cauchy-path $\mathcal{C}_n$ to $\mathcal{C}_m$ ) on the algebra $\overline{\mathcal{P}(Q_{\mathcal{C}_l})}$ is an algebra automorphism*

$$\phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n} : \overline{\mathcal{P}(Q_{\mathcal{C}_l})} \to \overline{\mathcal{P}(Q_{\mathcal{C}_l})}$$

*which preserves the *-operation and the commutator in the generators of the algebra, i.e.*

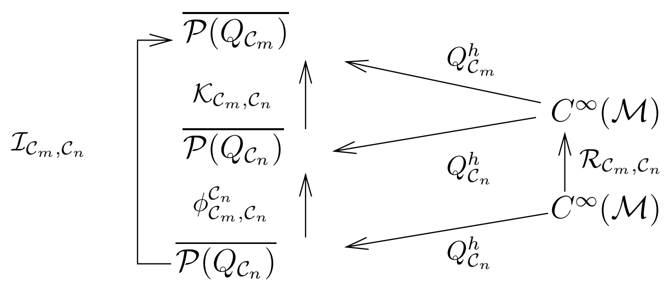
$$\begin{aligned} \phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}(a^*) &= \phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}(a)^* \qquad a \in \overline{\mathcal{P}(Q_{\mathcal{C}_l})} \\ \phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}(Q^{\mathcal{C}_l}_k Q^{\mathcal{C}_l}_j (Q^{\mathcal{C}_l}_k)^{-1} (Q^{\mathcal{C}_l}_j)^{-1}) &= Q^{\mathcal{C}_l}_k Q^{\mathcal{C}_l}_j (Q^{\mathcal{C}_l}_k)^{-1} (Q^{\mathcal{C}_l}_j)^{-1} \in \overline{\mathcal{P}(Q_{\mathcal{C}_l})} \end{aligned}$$

Clearly the conjugation with a invertible element in the algebra defines an evolution automorphism, this can be extended to the following:

**Definition 3.2.2** *An* **almost hamiltonian evolution automorphism** $\phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}$ *on the algebra $\overline{\mathcal{P}(Q_{\mathcal{C}_l})}$ is an evolution automorphism, which is constructed by conjugation with an invertible element $U^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n} \in \overline{\mathcal{P}(Q_{\mathcal{C}_l})}$ and multiplication with a phase factor $e^{i\eta^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}}$:*

$$\phi^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}(a) := (U^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n})^{-1}\, a\, U^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n} e^{i\eta^{\mathcal{C}_l}_{\mathcal{C}_m,\mathcal{C}_n}} \qquad a \in \overline{\mathcal{P}(Q_{\mathcal{C}_l})}$$

An evolution automorphism, may now serve as a quantized coordinate change (please refer to section 2.2), if it is compatible with quantization, i.e. if the following diagram is commutative:




The map $\mathcal{R}_{\mathcal{C}_m,\mathcal{C}_n} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ was defined in (2.2.6) as a coordinate change on phase space being a torus.

The map $\mathcal{K}_{\mathcal{C}_m,\mathcal{C}_n} : \overline{\mathcal{P}(Q_{\mathcal{C}_n})} \to \overline{\mathcal{P}(Q_{\mathcal{C}_m})}$ connects the quantized coordinate components along a Cauchypath $\mathcal{C}_m$, whose representation is dependend on the generators of $\overline{\mathcal{P}(Q_{\mathcal{C}_n})}$, with their representation as generators of $\overline{\mathcal{P}(Q_{\mathcal{C}_m})}$. The map $\mathcal{I}_{\mathcal{C}_m,\mathcal{C}_n}$ identifies the quantizations of the coordinates assigned to different Cauchy paths.

If two Cauchypaths $\mathcal{C}_n$ and $\mathcal{C}_m$ are isomorphic to each other then one would expect that the map $\mathcal{I}_{\mathcal{C}_m,\mathcal{C}_n}$ should be at least an algebraic isomorphism, i.e. it should preserve the algebraic structure like linear combinations, products and adjoints, which entails also topological isomorphism (cf. R. Haag, Local Quantum Physics 1992, p.115)[27]. If $\mathcal{I}_{\mathcal{C}_m,\mathcal{C}_n}$ is an isomorphism such that $\overline{\mathcal{P}(Q_{\mathcal{C}_m})} \cong \overline{\mathcal{P}(Q_{\mathcal{C}_n})} \cong \overline{\mathcal{P}(Q)}$ and one chooses a representation of the algebra $\overline{\mathcal{P}(Q)}$ on some Hilbert space, then the map $\mathcal{K}_{\mathcal{C}_m,\mathcal{C}_n}$ which mediates between operators at possibly equal space time points but with respect to different reference frames should be for physical reasons a change of basis on Hilbert space, i.e. a conjugation.

**Remark 3.2.3** *We note that if the evolution automorphism $\phi^{\mathcal{C}_n}_{\mathcal{C}_m,\mathcal{C}_n}$ is given explicitly on the algebra $\overline{\mathcal{P}(Q_{\mathcal{C}_n})}$ then this strongly restricts the available quantizations $Q^h_{\mathcal{C}_n}$, since*

$$Q^h_{\mathcal{C}_n}(\mathcal{R}_{\mathcal{C}_m,\mathcal{C}_n}(q^{\mathcal{C}_n})) := \phi^{\mathcal{C}_n}_{\mathcal{C}_m,\mathcal{C}_n}(Q^h_{\mathcal{C}_n}(q^{\mathcal{C}_n}))$$

## 3.3 Quantization of previously introduced models

The simplest space time one can think of, is the already in previous chapters discussed Minkowski space time lattice. We want to establish a quantum model on this space time by quantizing the discrete classical models, which were investigated in the last chapter.

### 3.3.1 Quantization of edge variables

In the appendix we learned (5.1) that there exists a two parameter family of poisson structures (5.1.11) on the algebra generated by the edgevariables $\{(u_{t,k})_{k\in\{0,\dots 2p\}}, (v_{t,k})_{k\in\{0,\dots 2p\}}\}$ which leads to an evolution compatible poisson structure on the algebra generated by face variables .

However in order to define a definite evolution on the algebra of edge variables, i.e. in order to define a phase space in the sense of section 2.2 it is necessary to fix a gauge, i.e. to fix the variables $c_{t,k}$ in:

$$\begin{aligned}
x_{t+1,k-1} &= V'(x_{t,k-1} - x_{t,k} - o_{t,k-1}) + x_{t,k-1} + c_{t-1,k-1} - c_{t+1,k-1} \qquad (3.3.3)\\
x_{t+1,k} &= V'(x_{t,k-1} - x_{t,k} - o_{t,k-1}) + x_{t,k} + c_{t-1,k-1} - c_{t+1,k-1}\\
w_{t+1,k-1} &= V'(-w_{t,k-1} - w_{t,k} + o_{t,k-1}) + w_{t,k-1} - (c_{t-1,k-1} - c_{t+1,k-1})\\
w_{t+1,k} &= -V'(-w_{t,k-1} - w_{t,k} + o_{t,k-1}) + w_{t,k} + c_{t-1,k-1} - c_{t+1,k-1}
\end{aligned}$$

If we choose $c_{t,k} = const.$ then among the poisson structures given by (5.1.11) only the case $b = 0, c = 0$ is compatible with this gauge fixing.

Besides this special gauge fixing we do not apriori know what kind of gauge fixings are compatible with the poisson structures given in (5.1.11), neither it is obvious wether all poisson structures in (5.1.11) can be interpreted as belonging to a certain constraint system obtained by a gauge fixing. We could also interpret the edge variables as representatives of an equivalence class obtained by the equivalence relation of gauge transformations. Nevertheless we have to proceed, hence for the moment the edge algebra together with its translational invariant poisson relations (5.1.11) will be viewed as functions on some kind of phase space and in this sense quantized by definition (3.1.7).

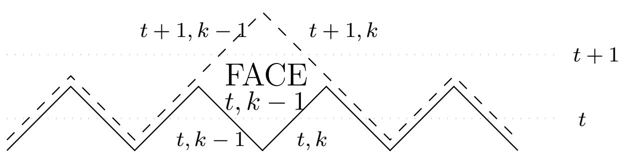
If we quantize the edge variables along an elementary Cauchy zig zag by the procedure given in (3.1.7) we obtain unitary operators which are labeled by the edges of our elementary Cauchy zig zag.

$$U_{t,k} := Q^h(e^{iu_{t,k}}), \qquad V_k := Q^h(e^{iv_{t,k}})$$

and which satisfy the commutation relations

$$U_{t,k}V_{t,j} = e^{ihc_{kj}}V_{t,j}U_{t,k}$$

where $c_{kj}$ is given by the classical commutation relations in (5.1.11). We abreviate from now on $q = e^{ih}$.
The generators $\{(U_{t,k})_{k\in\{0,...2p\}},(V_{t,k})_{k\in\{0,...2p\}}\}$ generate the $C^*$-algebra $\overline{\mathcal{P}(Q_{\mathcal{C}_t})}$ (see definition 3.1.4). Let $\mathcal{C}_{t,k-1}$ denote a Cauchy path which is obtained from an elementary Cauchy zig zag $\mathcal{C}_t$ by swapping an elementary "zag" at the face site $t, k-1$:




## 3.3.2 Dynamics for the quantized edge variables

We will now impose the following three demands for the definition of an evolution automorphism $\phi_{t,k-1} := \phi^{\mathcal{C}_t}_{\mathcal{C}_{t,k-1},\mathcal{C}_t}$ which shall describe the time evolution of the edge operator algebra at the Cauchy path $\mathcal{C}_t$ to the Cauchy path $\mathcal{C}_{t,k-1}$.

**Demands for time step $\phi_{t,k-1}$**

a. $\phi_{t,k-1}(a_{t,j}) = a_{t,j}$ if $j \neq k-1 \wedge j \neq k \, mod \, 2p \quad a = U, V$

(edge operators associated to edges which are not part of the (t,k-1)'th face won't be evolved)

b. $\phi_{t,k-1}(P_{t,j}) = P_{t,j} \qquad \phi_{t,k-1}(O_{t,j}) = O_{t,j}$,

(face operators associated to the (t,k-1)'th face stay the same) where

$$P_{t,k-1} := V^{-1}_{t,k}V_{t,k-1}U^{-1}_{t,k-1}U^{-1}_{t,k} \tag{3.3.4}$$

$$O_{t,k-1} := V^{-1}_{t,k}V_{t,k-1}U_{t,k}U_{t,k-1} \tag{3.3.5}$$

c. The quantum zero curvature condition (QZC)

$$L_{t+1,k}L_{t+1,k-1} = L_{t,k}L_{t,k-1} \tag{3.3.6}$$

shall be satisfied, with

$$L_{t,k} = L_{t,k}(\lambda + \omega_{t,k}) = \begin{pmatrix} U_{t,k} & -e^{\lambda+\omega_{t,k}}V_{t,k}^{-1} \\ e^{\lambda+\omega_{t,k}}V_{t,k} & U_{t,k}^{-1} \end{pmatrix}$$

and $\omega_{t,k} = (-1)^i \omega_j^i$, $\omega_{t,k} \in \mathbb{R}$ where

$$\begin{array}{llll} i = 1 & , & j = k - t & \text{if } k - t \text{ odd} \\ i = 2 & , & j = k + t & \text{if } k - t \text{ even} \end{array}$$

Note that the classical matrix $L_{t,k}$ was already introduced in 1.3.28.

### Construction of time step $\phi_{t,k-1}$

In order to ensure the validity of the quantum zero curvature condition (QZC), which was stated in demand c), we will first define an evolution for the above quantum matrices $L_{t,k}$, which in the turn induces an evolution automorphism $\phi_{t,k-1}$ on the edge algebra.

Suppose we are given an initial configuration of matrices $L_{t,k}(\lambda + \omega_{t,k})$ for a fixed time $t$. Define recursively:

$$L_{t+1,k}(\lambda + \underbrace{\omega_{t+1,k}}_{=\omega_{t,k-1}}) := \tilde{R}L_{t,k}(\lambda + \omega_{t,k-1})R^{-1} \tag{3.3.7}$$

$$L_{t+1,k-1}(\lambda + \underbrace{\omega_{t+1,k-1}}_{=\omega_{t,k}}) := RL_{t,k-1}(\lambda + \omega_{t,k})\tilde{R}^{-1} \tag{3.3.8}$$

where $R$ and $\tilde{R}$ haven't been specified yet and $L_{t,k}(\lambda + \omega_{t,k-1})$ is to be obtained from the given initial matrix $L_{t,k}(\lambda + \omega_{t,k})$ by replacing $\omega_{t,k}$ with $\omega_{t,k-1}$. In particular we assume that we know the values of the $\omega_{t,k}$ for all times and positions. The variables $\omega_{t,k}$ will be here not treated as dynamical variables.

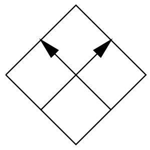
**Remark 3.3.1** *The matrix $L_{t,k}(\lambda+\omega_{t,k})$ can also be obtained by defining "shifts" $S_a$ on the edge algebra, i.e. automorphisms which map a generator of the edgealgebra onto a generator originally assigned to a neighbouring edge, i.e.*

$$S_{A_{t,k}}(A_{t,k}) = A_{t,k+1} \quad A = U, V$$

*and then extending this construction to a "shift" on the algebra of initial L-matrices $L_{t,k}(\lambda + \omega_{t,k})$ :*

$$L_{t,k}(\lambda+\omega_{t,k-1}) = S_{t,k-1}(L_{t,k-1}) := \begin{pmatrix} S_{U_{t,k}}(U_{t,k}) & -e^{\lambda+\omega_{t,k-1}}S_{V_{t,k}}(V_{t,k}^{-1}) \\ e^{\lambda+\omega_{t,k-1}}S_{V_{t,k}}(V_{t,k}) & S_{U_{t,k}}(U_{t,k}^{-1}) \end{pmatrix}.$$

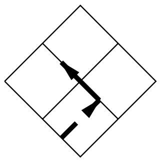
*With such a construction the recursive definition of the matrices in (3.3.8) can be interpreted as a "propagation in light cone direction":*



*Such a viewpoint was taken in [52]. Here the generators* $A_{t,k}$ *were chosen to be ultralocally commuting (case* $b = 0$, $c = 2a$ *in (5.1.11)) and could be interpreted as operators which act nontrivially only on a local Hilbertspace* $\mathcal{H}_{t,k}$ *(assigned to the k'th edge). Hence if we form a big Hilbertspace* $\mathcal{H}_t = \bigotimes_k \mathcal{H}_{t,k}$ *which is assigned to the whole Cauchy path* $\mathcal{C}_t$*, then* $A_{t,k} := \mathbb{1} \otimes \mathbb{1} \otimes \ldots \underbrace{A}_{k'thsite} \ldots \otimes \mathbb{1} \otimes \mathbb{1}$ *,*
*where* $\mathbb{1}$ *is the identity on* $\mathcal{H}_{t,k}$*. The above shift* $S_A$ *in this context is then a global shift in the tensorproduct of operators:*

$$S^{\pm 1}(A_{t,k}) := \mathbb{1} \otimes \mathbb{1} \otimes \ldots \underbrace{A}_{(k\pm 1)'thsite} \ldots \otimes \mathbb{1} \otimes \mathbb{1} \,.$$

*Moreover in this work the propagation in light cone direction was not only defined along a face (dashed arrow), but in the same manner as above also defined* between *faces (ligned arrow) (please confer [52]):*



*Although this definition is very suggestive it unfortunately imposes strong restrictions on the element R in order to ensure the commutativity of the lightcone evolutions. In particular the element R has to be the same for all faces. In a later section the notion of a space shift will be generalized.*

For our purpose we would like to define an element $\tilde{R} = \tilde{R}_{t,k-1}$ where the subindex $(t, k-1)$ indicates the evolution from Cauchypath $\mathcal{C}_t$ to Cauchypath $\mathcal{C}_{t,k-1}$. In particular this implies that the needed $\tilde{R}$-operator shall depend explicitly on the chosen face. We make the following ansatz:

$$\tilde{R}(P_{t,k-1}) := \begin{pmatrix} e^{i\eta(P_{t,k-1})}\mathbb{1} & 0 \\ 0 & e^{-i\eta(P_{t,k-1})}\mathbb{1} \end{pmatrix} \underbrace{\begin{pmatrix} R(P_{t,k-1}) & 0 \\ 0 & R(P_{t,k-1}) \end{pmatrix}}_{=:\, R(P_{t,k-1})} \tag{3.3.9}$$

where $e^{i\eta(P_{t,k-1})} \in S^1 \subset \mathbb{C}$ is a number depending on the chosen face operator $P_{t,k-1}$ and $R(P_{t,k-1})$ is a Laurent polynomial in the face operator $P_{t,k-1}$. Note that since $R(P_{t,k-1})$ is polynomial in the face operator $P_{t,k-1}$ it follows immediately that if $A$ is an operator, which commutes with $P_{t,k-1}$ as $P_{t,k-1}A = q^x A P_{t,k-1}$, $q^x \in S^1$ then

$$R(P_{t,k-1})A = AR(q^x P_{t,k-1}).$$

Definition (3.3.9) induces immediately the following evolution automorphism on the edgealgebra (k-t even):

$$\begin{aligned} U_{t+1,k} &= R(P_{t,k-1})U_{t,k}R(P_{t,k-1})^{-1}e^{i\eta(P_{t,k-1})} \\ U_{t+1,k-1} &= R(P_{t,k-1})U_{t,k-1}R(P_{t,k-1})^{-1}e^{-i\eta(P_{t,k-1})} \\ V_{t+1,k} &= R(P_{t,k-1})V_{t,k}R(P_{t,k-1})^{-1}e^{-i\eta(P_{t,k-1})} \\ V_{t+1,k-1} &= R(P_{t,k-1})V_{t,k-1}R(P_{t,k-1})^{-1}e^{-i\eta(P_{t,k-1})} \end{aligned} \tag{3.3.10}$$

Edge operators at different sites won't be evolved (mod 2p). By (3.3.11) the edge operators at the considered face shall be evolved via a conjugation with an operator $R(P_{t,k-1}) \in \overline{\mathcal{P}(Q_{\mathcal{C}_t})}$ and multiplication with a factor. Considering the commutation relations given in (5.1.11) and the so induced commutation relations on the quantum operators, we see that beside demand a) also demand b) is automatically satisfied by the above ansatz of an evolution automorphism (3.3.9, 3.3.11). Demand c) results in a condition on the operator $\tilde{R}$, namely:

$$\begin{aligned} L_{t+1,k}(\lambda + \underbrace{\omega_{t+1,k}}_{\omega_{t,k-1}})L_{t+1,k-1}(\lambda + \underbrace{\omega_{t+1,k-1}}_{\omega_{t,k}}) &\overset{3.3.8}{:=} \tilde{R}L_{t,k}(\lambda + \omega_{t,k-1})L_{t,k-1}(\lambda + \omega_{t,k})\tilde{R}^{-1} \\ &\overset{!}{=} L_{t,k}(\lambda + \omega_{t,k})L_{t,k-1}(\lambda + \omega_{t,k-1}). \end{aligned}$$

The equation

$$\tilde{R}L_{t,k}(\lambda + \omega_{t,k-1})L_{t,k-1}(\lambda + \omega_{t,k}) = L_{t,k}(\lambda + \omega_{t,k})L_{t,k-1}(\lambda + \omega_{t,k-1})\tilde{R} \tag{3.3.11}$$

will be called a generalized Yang Baxter equation (GYBE) since it can be shown that $\tilde{R}$ and $L$ can sometimes be interpreted as representations of a socalled universal R-matrix [1] which satisfies the Yang-Baxter equation appearing in the theory of quantum groups (cf. eg [25]). In the applications we are having in mind the considered quantum group is usually $U_q(\hat{sl}_2)$. (cf. e. g. [52]).
Evaluating (3.3.11) we obtain the following independent equations (k-t even):

$$R(P_{t,k-1})U_{t,k}U_{t,k-1}R^{-1}(P_{t,k-1}) = U_{t,k}U_{t,k-1} \tag{3.3.12}$$

$$R(P_{t,k-1})V_{t,k}^{-1}V_{t,k-1}R^{-1}(P_{t,k-1}) = V_{t,k}^{-1}V_{t,k-1} \tag{3.3.13}$$

$$\begin{aligned} e^{2i\eta(P_{t,k-1})}R(P_{t,k-1})e^{\omega^2_{k+t}}U_{t,k}V_{t,k-1}^{-1} &+ e^{-\omega^1_{k-t-1}}V_{t,k}^{-1}U_{t,k-1}^{-1}R^{-1}(P_{t,k-1}) = \\ e^{-\omega^1_{k-t-1}}U_{t,k}V_{t,k-1}^{-1} &+ e^{\omega^2_{k+t}}V_{t,k}^{-1}U_{t,k-1}^{-1} \end{aligned} \tag{3.3.14}$$

All other equations to be obtained from (3.3.11) follow from (3.3.12-3.3.14).

By the commutation relations given in (5.1.11) and the so induced commutation relations on the quantum operators, we see that equations (3.3.12) and

(3.3.13) are automatically satisfied by the above ansatz, in fact they are equivalent to demand b). So we are left to investigate equation (3.3.14). We obtain the following functional equation:

$$e^{2i\eta(P_{t,k-1})}\frac{R(P_{t,k-1})}{R(q^{8a}P_{t,k-1})} = \frac{e^{-\omega^2_{k+t}+\omega^1_{k-t-1}} + q^{2a+c}P_{t,k-1}}{1+e^{-\omega^2_{k+t}+\omega^1_{k-t-1}}q^{2a+c}P_{t,k-1}} \tag{3.3.15}$$

If $|q| < 1$ and $\eta = 0$ an explicit solution of this equation can be found in [52]. We are interested in the case for which $|q| = 1$, where only for special cases solutions were constructed in the literature.

### Construction of solutions to the GYBE at roots of unity

For a further analysis of equation (3.3.15) we need the following Lemma:

**Lemma 3.3.2** *Let $m < N \in \mathbb{N}$ and $q$ be a root of unity, i.e. $q = e^{2i\pi/N}$. Let $gcd(N, m)$ be the greatest common divisor of $N$ and $m$. If $B := \frac{N}{gcd(m,N)}$ then $q^{mB} = 1$ and $q^{mA} \neq 1$ for all $A \in \mathbb{N}$, $A < B$.*

**Proof:** We look for a number $B \in \mathbb{N}$, $B \leq N$ with $q^{mB} = 1$, where $m \in \mathbb{N}$, $m < N$ such that for all $A \in \mathbb{N}$, $A < B$; $q^{mA} \neq 1$. We know that there exists a number $t \in \mathbb{N}$ such that

$$Bm = tN.$$

Define

$$a := \frac{m}{gcd(m, N)} \qquad b := \frac{N}{gcd(m, N)}$$

Since $m < N$, $a \neq b$ and $gcd(a, b) = 1$. So it follows from

$$Bm = tN \Leftrightarrow Ba = tb \quad \Rightarrow \quad B = b \quad \wedge \quad a = t.$$

□

**Corollary 3.3.3** *If $B := \frac{N}{gcd(m,N)}$, $m < N \in \mathbb{N}$ and $q = e^{2i\pi/N}$; if $A_1, A_2 \in \mathbb{N}, A_2 > A_1, A_i < B$ then $q^{(A_2-A_1)m} \neq 1$ or in other words all roots of unity $q^{A_i m}$ for which $A_i < B$ are all different from each other.*

**Proposition 3.3.4** *Let $N \in \mathbb{N}$, $q = e^{2i\pi/N}$, $B = \frac{N}{gcd(m,N)}$, $m = 2$ then the following identity holds:*

$$\prod_{j=0}^{B-1} \frac{k + q^{jm}x}{1 + q^{jm}kx} = \frac{k^B + x^B(-1)^{B-1}}{1 + k^B x^B (-1)^{B-1}} \tag{3.3.16}$$

**Proof:** All roots of the RHS denominator and nominator polynomials are only assumed once because of Corollary (3.3.3). Hence the LHS is nothing else then splitting the denominator and nominator polynomials of the RHS into their linear factors. So poles and zeros of LHS and RHS are the same, checking the equality at e.g. $x = 0$ gives the assertion.
□

**Theorem 3.3.5** *Let $\hat{x}$ be an element of a $C^*$-algebra, such that $x^{-1} = x^*$ and $\hat{x}^B := \underbrace{\hat{x} \cdot \ldots \cdot \hat{x}}_{B \text{ times}}$ is a multiple of the identity element $\mathbb{1}$ in the algebra, i.e. $\hat{x}^B = x^B \mathbb{1}$, with $x^B \in S^1 \subset \mathbb{C}$. Let $k \in [0,1), B = \frac{N}{gcd(m,N)}, q = e^{\frac{2\pi i}{N}}$. Choose any root*

$$e^{i\xi_k} := \left( \frac{1 + k^B x^B (-1)^{B-1}}{k^B + x^B (-1)^{B-1}} \right)^{\frac{1}{B}} \quad \in S^1. \tag{3.3.17}$$

*Define $R_k(\hat{x}) := \sum_{j=0}^{B-1} l_j \hat{x}^j$ with*

$$l_j := \prod_{n=1}^{j} \frac{e^{i\xi_k} q^{m(n-1)} - k}{1 - e^{i\xi_k} k q^{mn}}$$

*then $R_k(\hat{x})$ satisfies the functional equation:*

$$\frac{R_k(\hat{x})}{R_k(q^m \hat{x})} = \frac{k + \hat{x}}{1 + k\hat{x}} e^{i\xi_k}.$$

**Proof:** Direct verification under the use of proposition (3.3.4).
□
**Remark to the Notation***If $R(x)$ appears in the text without an index $k$, then $R(x) = R(x)_k$, where $k$ is arbitrary*
An explicit solution to equation (3.3.11) for the root of unity case was originally constructed for $\xi = 0$, $a = \frac{1}{4}$, $N = odd$ in [54] and [23]. Besides the already mentioned case where $|q| < 1$ an asymptotic description of the solution $R$ for $|q| = 1$ and $\xi \neq 0$ were given in [52]. The construction of the above solutions is to our knowledge new.

**Lemma 3.3.6** *Let $q = e^{\frac{2\pi i}{N}}$,$N \in 2\mathbb{N} + 1$, $\tilde{R}, \xi, \hat{x}^N$ as before in theorem (3.3.5). Let*

$$f(\hat{x}) = \frac{k + \hat{x}}{1 + k\hat{x}} e^{i\xi_k}$$

*then*

$$\frac{R_k(\hat{x})}{R_k(q\hat{x})} = e^{i\xi_k (N+1)/2} \prod_{j=0}^{\frac{N-1}{2}} f(q^{2j}\hat{x}) \tag{3.3.18}$$

**Proof:**
We make the ansatz

$$\frac{R_k(\hat{x})}{R_k(q\hat{x})} = \alpha(\hat{x}) \prod_{j=0}^{\frac{N-1}{2}} f(q^{2j}\hat{x})$$

where $\alpha(\hat{x})$ shall be some polynomial in $\hat{x}$ then

$$e^{i\xi_k} f(\hat{x}) = \frac{R_k(\hat{x})}{R_k(q\hat{x})} \frac{R_k(q\hat{x})}{R_k(q^2\hat{x})} \stackrel{!}{=} \alpha(\hat{x})^2 \prod_{j=0}^{\frac{N-1}{2}} f(q^{2j}\hat{x}) f(q^{2j+1}\hat{x}) = \alpha(\hat{x})^2 e^{-i\xi_k N} f(\hat{x})$$

because of proposition (3.3.4). Hence $\alpha(\hat{x})^2 = \alpha(\hat{x})^2 = e^{i\xi_k(N+1)}\mathbb{1}$ .
y□

**Corollary 3.3.7** *Let $q$ be a root of unity, i.e. $q = e^{\frac{2\pi i}{N}}$ and the notations as in theorem (3.3.5). If $\tilde{R}$ is invertible, if $P^B_{t,k-1}$ can be chosen as a multiple of the identity in $\overline{\mathcal{P}(Q_{\mathcal{C}_t})}$ and if the conjugation with $R$ maps unitary operators into unitary operators then the evolution automorphism $\phi_{t,k-1}$ defined in (3.3.11) and (3.3.9) with the R-matrix as in theorem (3.3.5) is compatible with the quantization given in (3.1.7) and satisfies demands a)-c).*

**Proof:**
Demands a)-c) are satisfied by the explicit construction of the matrix $\tilde{R}$. The evolved operators which are obtained by applying the evolution automorphism to generators of the edge algebra are again elements of $\overline{\mathcal{P}(Q_{\mathcal{C}_t})}$ since the conjugation with a polynomial in the generators of $\overline{\mathcal{P}(Q_{\mathcal{C}_t})}$ and incident multiplication with a number is an automorphism in $\overline{\mathcal{P}(Q_{\mathcal{C}_t})}$.

Moreover it is straightforward to check that condition 1.)-6.) of (3.1.7) do not conflict with the definition of evolved operators.
□

Clearly $P^B_{t,k-1}$ cannot always be chosen to be a multiple of the identity, yet if $N = odd$, $a = \frac{1}{4}$, $b = 0$ then this is always possible:

**Corollary 3.3.8** *Let $q = e^{\frac{2\pi i}{N}}$,$N \in 2\mathbb{N}+1$, $\phi_{t,k-1}$ as defined before and the commutation relations in (5.1.11) such that $a = \frac{1}{4}$ and $b = 0$. Then the map $\phi_{t,k-1}$ is an evolution automorphism which is compatible with quantization.*

**Proof:**
The proof that $\tilde{R}$ is invertible will be postponed. The commutation relations between edgeoperators and face variables are given by (5.1.11):

$$\begin{aligned}
U_{t,k}P_{t,k-1} &= q^{4a+b}P_{t,k-1}U_{t,k} \\
U_{t,k-1}P_{t,k-1} &= q^{-4a-b}P_{t,k-1}U_{t,k-1} \\
V_{t,k}P_{t,k-1} &= q^{-4a+b}P_{t,k-1}V_{t,k} \\
V_{t,k-1}P_{t,k-1} &= q^{4a-b}P_{t,k-1}V_{t,k-1}
\end{aligned} \tag{3.3.19}$$

and

$$[A_{t,k}, P_{t,k+j}] = 0, \quad [A_{t,k}, P_{t,k+j}] = 0$$

for $j \in 1...2p-2$ and $A = U, V$, where $2p$ is the period of the face variables. Hence if $b = 0$ and $a = \frac{1}{4}$ we see that by lemma 3.3.6 that the unitary edge operators are mapped into unitary operators since the function $f(x) = \frac{k+x}{1+kx}$ is an automorphism of the unit circle. $P^B_{t,k-1} = P^N_{t,k-1}$ is a casimir in this algebra and hence can be chosen to be a multiple of the identity.
□

Hence we have constructed an explicit example for an evolution automorphism $\phi_{t,k-1}$ on the edge algebra.

**Remark 3.3.9** *Let us consider again the root of unity case with $b = 0$ of the commutation relations given in (5.1.11) By using these commutation relations (compare also with (3.3.19)) we note that although $P^B_{t,k-1}$ need not to commute trivially with the edge operators $U_{t,j}$, $V_{t,j}$ assigned to the edges which align the face $(t, k-1)$ it will commute trivially within the subalgebra generated by equal time pairs $A_{t,j}A_{t,l}$, $A = U, V$ of edge variables assigned to these edges. In particular the evolution for these "paired edgevariables" $A_{t,j}A_{t,l}$, $A = U, V$ defined by the formal application of the evolution automorphism $\phi_{t,k-1}$ will be welldefined as well as the quantum zero curvature condition - regardless of the chosen root of unity. The proof of this assertion was basically given in (3.3.5) and by the commutation relations in (5.1.11).*

An example of an evolution for such "paired edge variables" will be given in the next section.

### 3.3.3 An evolution equation for squared edge operators

Although if the evolution automorphism $\phi_{t,k-1}$ is not necessarily well defined on the algebra generated by the edge variables the quantum zero curvature condition (QZC) will still hold (compare remark 3.3.9) and will give us the following three independent equations:

$$U_{t+1,k}U_{t+1,k-1} = U_{t,k}U_{t,k-1} \tag{3.3.20}$$

$$V^{-1}_{t+1,k}V_{t+1,k-1} = V^{-1}_{t,k}V_{t,k-1} \tag{3.3.21}$$

$$\begin{aligned} e^{\omega^2_{k+t}}U_{t+1,k}V^{-1}_{t+1,k-1} + e^{-\omega^1_{k-t-1}}V^{-1}_{t+1,k}U^{-1}_{t+1,k-1} = \\ e^{-\omega^1_{k-t-1}}U_{t,k}V^{-1}_{t,k-1} + e^{\omega^2_{k+t}}V^{-1}_{t,k}U^{-1}_{t,k-1} \end{aligned} \tag{3.3.22}$$

A careful analysis shows that the first two equations (3.3.20,3.3.21) are identical to demands a) and b). Analogously to the classical case we will now construct an evolution equation in the edge variables by rewriting equation (3.3.22). Define

$E_{t,k-1} := U_{t,k-1}^{-1} V_{t,k-1} U_{t,k} V_{t,k}$. By the definition of the evolution automorphism (3.3.11) we find:

$$E_{t+1,k-1} := \frac{R_k(P_{t,k-1})}{R_k(q^{4b} P_{t,k-1})} E_{t,k-1}$$

That means that if $b = 0$ in (5.1.11) then, on a face, the operator $E_{t+1,k-1} E_{t,k-1}^{-1}$ is the identity in the algebra $\overline{\mathcal{P}(Q_{\mathcal{C}_t})}$. Comparing with the classical case we find that the variable $E_{t+1,k-1} E_{t,k-1}^{-1}$ defines the evolution of the gauge fixing variables $c_{t,k}$ along the face $(t, k-1)$ (confer (2.4.57)) . Using the identities (3.3.20,3.3.21) we see that the "quantum square root" of the expression $E_{t+1,k-1} E_{t,k-1}^{-1}$ $= V_{t+1,k-1}^2 U_{t+1,k}^2 U_{t,k}^{-2} V_{t,k-1}^{-2}$ (use (3.3.20),(3.3.21)) is given by

$$F_{t,k-1} := V_{t+1,k-1} U_{t+1,k} U_{t,k}^{-1} V_{t,k-1}^{-1} = \frac{R(P_{t,k-1})}{R(q^{2b} P_{t,k-1})}. \tag{3.3.23}$$

Hence the determination of $F_{t,k-1}$ gives us the gauge constraint which is necessary to construct a definite evolution from the QZC.
Now rewriting equation (3.3.22) we get:

$$U_{t+1,k} V_{t+1,k-1}^{-1} V_{t,k-1} U_{t,k}^{-1} = \frac{e^{-\omega_{k+t}^2 + \omega_{k-t-1}^1} + q^{2a+c} P_{t,k-1}}{1 + e^{-\omega_{k+t}^2 + \omega_{k-t-1}^1} q^{2a+c} P_{t,k-1}} \tag{3.3.24}$$

Using the definition of $F_{t,k-1}$ we obtain

$$\begin{aligned} U_{t+1,k} V_{t+1,k-1}^{-1} V_{t,k-1} U_{t,k}^{-1} &= V_{t+1,k-1}^{-2} \frac{R(P_{t,k-1})}{R(q^{2b} P_{t,k-1})} V_{t,k-1}^2 \\ &= V_{t+1,k}^{-2} \frac{R(P_{t,k-1})}{R(q^{2b} P_{t,k-1})} V_{t,k}^2 \\ &= U_{t+1,k}^2 \frac{R(P_{t,k-1})}{R(q^{-2b} P_{t,k-1})} U_{t,k}^{-2} \\ &= U_{t+1,k-1}^{-2} \frac{R(P_{t,k-1})}{R(q^{-2b} P_{t,k-1})} U_{t,k-1}^2 \end{aligned}$$

Furthermore (see also definition (3.3.5))

$$\begin{aligned} P_{t,k-1} &= V_{t,k}^{-2} V_{t,k-1}^2 O_{t,k-1}^{-1} q^{-2a+b+c} && (3.3.25) \\ &= U_{t,k-1}^{-2} U_{t,k}^{-2} O_{t,k-1} q^{-2a-b+c}. && (3.3.26) \end{aligned}$$

We obtain the evolution equations (k-t even)

$$V_{t+1,k-1}^{-2} = \frac{e^{-\omega_{k+t}^2 + \omega_{k-t-1}^1} + q^{b+2c} V_{t,k}^{-2} V_{t,k-1}^2 O_{t,k-1}^{-1}}{1 + e^{-\omega_{k+t}^2 + \omega_{k-t-1}^1} q^{b+2c} V_{t,k}^{-2} V_{t,k-1}^2 O_{t,k-1}^{-1}} V_{t,k-1}^{-2} \frac{R(q^{2b} P_{t,k-1})}{R(P_{t,k-1})} \tag{3.3.27}$$

$$\begin{aligned} V^{-2}_{t+1,k} &= \frac{e^{-\omega^2_{k+t}+\omega^1_{k-t-1}} + q^{b+2c}V^{-2}_{t,k}V^2_{t,k-1}O^{-1}_{t,k-1}}{1+e^{-\omega^2_{k+t}+\omega^1_{k-t-1}}q^{b+2c}V^{-2}_{t,k}V^2_{t,k-1}O^{-1}_{t,k-1}}V^{-2}_{t,k}\frac{R(q^{2b}P_{t,k-1})}{R(P_{t,k-1})} \\ U^2_{t+1,k} &= \frac{e^{-\omega^2_{k+t}+\omega^1_{k-t-1}} + q^{-b+2c}U^{-2}_{t,k-1}U^{-2}_{t,k}O_{t,k-1}}{1+e^{-\omega^2_{k+t}+\omega^1_{k-t-1}}q^{-b+2c}U^{-2}_{t,k-1}U^{-2}_{t,k}O_{t,k-1}}U^2_{t,k}\frac{R(q^{-2b}P_{t,k-1})}{R(P_{t,k-1})} \qquad (3.3.28) \\ U^{-2}_{t+1,k-1} &= \frac{e^{-\omega^2_{k+t}+\omega^1_{k-t-1}} + q^{-b+2c}U^{-2}_{t,k-1}U^{-2}_{t,k}O_{t,k-1}}{1+e^{-\omega^2_{k+t}+\omega^1_{k-t-1}}q^{-b+2c}U^{-2}_{t,k-1}U^{-2}_{t,k}O_{t,k-1}}U^{-2}_{t,k-1}\frac{R(q^{-2b}P_{t,k-1})}{R(P_{t,k-1})}. \end{aligned}$$

where the face variable $O_{t,k-1}$ evolves as :

$$O_{t+1,k} = \frac{R(P_{t,k-1})}{R(q^{2b}P_{t,k-1})}\frac{R(P_{t,k-1})}{R(q^{-2b}P_{t,k-1})}O_{t-1,k}. \qquad (3.3.29)$$

We see that if the commutation relations in (5.1.11) are such that $b = 0$ then we have a unique evolution within the above squared edgealgebra generated by the operators $\{(U^2_{t,k})_{k\in\{0,...2p\}},\ (V^2_{t,k})_{k\in\{0,...2p\}}\}$. The operators $O_{t,k-1}$ are in that case quantum integrals of motion. We remember that in the case where the commutation relations in (5.1.11) were such that $b = 0, c = 0$ the operators $O_{t,k-1}$ were even casimirs in the edgealgebra, i.e. they were commuting with any element in the algebra given by the operators $\{(U_{t,k})_{k\in\{0,...2p\}},\ (V_{t,k})_{k\in\{0,...2p\}}\}$ . In a later section we will construct an explicit representation for the nonultralocal commuting squared edgeoperators (case $b = 0, c = 0$ in (5.1.11) and show how they can be related to a fermionic model.
In the next section another example for the evolution of "paired" edge variables will be given.

### 3.3.4 An evolution equation for vertex operators

By the definition of the face operator $P_{t,k-1}$ we see that

$$P_{t,k-1} = V^{-1}_{t,k}V_{t,k-1}U^{-1}_{t,k-1}U^{-1}_{t,k} = (V_{t,k-1}U^{-1}_{t,k-1})(V^{-1}_{t,k}U^{-1}_{t,k})q^{2a-b-c}$$

can be written in terms of edgeoperators which consist of products of $U$ and $V$-operators at one edge. Likewise we see that the same holds for the LHS of (3.3.24):

$$\begin{aligned} U_{t+1,k}V^{-1}_{t+1,k-1}V_{t,k-1}U^{-1}_{t,k} &= (U_{t+1,k}V^{-1}_{t+1,k})\underbrace{(V_{t,k}U^{-1}_{t,k})}_{=:A^{-1}_{t,k}} \qquad (3.3.30) \\ &= (V^{-1}_{t+1,k-1}U^{-1}_{t+1,k-1})\underbrace{(U_{t,k-1}V_{t,k-1})}_{=:B^{-1}_{t,k-1}} \qquad (3.3.31) \end{aligned}$$

Clearly the quantum zero curvature condition doesn't define a unique evolution for this new defined edgeoperators $A$ and $B$ (k-t even):

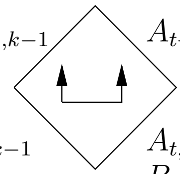
$$A_{t+1,k}B^{-1}_{t+1,k-1} = A_{t,k}B^{-1}_{t,k-1} = O_{t,k-1} \quad (3.3.32)$$
$$A^{-1}_{t+1,k-1}B_{t+1,k} = A^{-1}_{t,k-1}B_{t,k} = P_{t,k-1}q^{-2a+b+c} \quad (3.3.33)$$

$$A_{t+1,k}A^{-1}_{t,k} = B_{t+1,k-1}B^{-1}_{t,k-1} = f(A^{-1}_{t,k-1}B_{t,k}q^{2a-b-c}) \quad (3.3.34)$$

where

$$f(x) = \frac{k_{t,k-1} + q^{2a+c}x}{1 + k_{t,k-1}q^{2a+c}x} \quad \text{and} \quad k_{t,k-1} = e^{-\omega^2_{k+t}+\omega^1_{k-t-1}} \quad (3.3.35)$$

Or in other words if we are given $A_{t,k-1}$, $A_{t,k}$, $B_{t,k-1}$ and $B_{t,k}$ assigned to the $(t, k-1)$'th face then we obtain in a unique way the evolved operators $A_{t+1,k}$ and $B_{t+1,k-1}$ but not the operators $A_{t+1,k-1}$ or $B_{t+1,k}$. $A_{t+1,k-1}$ or $B_{t+1,k}$ can only be obtained by applying our definite evolution automorphism, which refers implicitly to a gauge fixing (confer also last section).




We note that we can rewrite (3.3.33) and (3.3.34) by using (3.3.33) into an evolution equation for the operators $A_{t,k}$ only:

$$A^{-1}_{t+1,k-1}O_{t+1,k}A_{t+1,k+1} = A^{-1}_{t,k-1}O_{t-1,k}A_{t,k+1}$$

$$A_{t+1,k} = f(A^{-1}_{t,k-1}O_{t-1,k}A_{t,k+1}q^{2a-b-c})A_{t,k}$$

As we found out in the previous section the operators $O_{t,k-1}$ are integrals of motion if the commutation relation in (5.1.11) are such that $b = 0$. The evolution of the operator $A_{t,k-1}$ (k-t even) reads in terms of the evolution automorphism as:

$$A_{t+1,k-1} = U_{t+1,k-1}V^{-1}_{t+1,k-1} = \frac{R(P_{t,k-1})}{R(q^{-2b}P_{t,k-1})}A_{t,k-1}$$

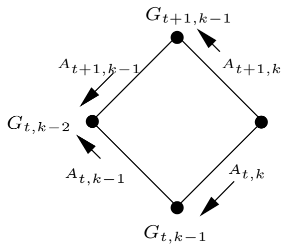
Hence if we assume that the commutation relations in (5.1.11) are such that $b = 0$ then we obtain the following picture of the evolution of the "paired edgeoperators" $A_{T,K}$ ($T, K$ arbitrary):

$$\begin{array}{rcll} A_{T+1,K} &=& A_{T,K} & \text{if} \quad K-T \quad \text{odd} \\ A_{T+1,K} &=& f(A^{-1}_{T,K-1}O_K A_{T,K+1})A_{T,K} & \text{if} \quad K-T \quad \text{even} \end{array} \quad (3.3.36)$$

where $O_K$ is a constant operator which is given by the initial operators $O_K = V^{-1}_{T_0,K+1}V_{T_0,K}U_{T_0,K+1}U_{T_0,K}$ where $T_0$ shall be the time ($K-T_0$ even) at which the evolution starts. Hence by the above we see immediately that the edgeoperators $A_{T,K}$ can be treated as vertex variables if one assigns to the edgeoperator, which is hooked to a certain edge, the vertex which lies to the left of the corresponding edge on the Cauchy zig zag under consideration. If the commutation relations are such that $b = 0$,$c = 2a$ (i.e. ultralocal) then the variables $A_{T,K}$ commute with each other trivially but nontrivially with the operators $O_K$. If the commutation relations are such that $b = 0$, $c = 0$ then the operator $O_K$ is a casimir in the algebra generated by the operators $\{(U_{t,k})_{k\in\{0,...2p\}},\ (V_{t,k})_{k\in\{0,...2p\}}\}$ , i.e. it can be treated as a number. In that case the variables $B_{T,K-1}$ are identical to the operator $A_{T,K}$ modulo the "factor" $O_{T,K-1}$. The operators $A_{T,K}$ commute like quantized vertex operators whose classical counterparts were introduced in section 2.3. Hence for that case it makes sense to define operators which are now assigned to the vertices of the lattice ("vertex operators"):

$$G_{t,k-1} := A_{t,k-1} = A_{t+1,k-1} \tag{3.3.37}$$

$$G_{t-1,k-1} := A_{t,k} \tag{3.3.38}$$




The evolution of the vertex variables $G_{t,k}$ is then given by (3.3.36) and reads as:

$$G_{t+1,k-1} = f(G^{-1}_{t,k-2}G_{t,k}O_K)G_{t-1,k-1} \tag{3.3.39}$$

which is modulo a redefinition along the diagonals of the space time lattice and of the additional parameter $O_K$ Hirotas vertex equation in quantized form.

### 3.3.5 An evolution equation for face operators

As we already found out in (3.3.30) and (3.3.31) the face variables are also expressable in terms of paired edge operators hence using (3.3.24) and (3.3.30-3.3.31) we can rewrite equation (3.3.14) as

$$U_{t+1,k}V^{-1}_{t+1,k} = f(P_{t,k-1})U_{t,k}V^{-1}_{t,k}$$

or

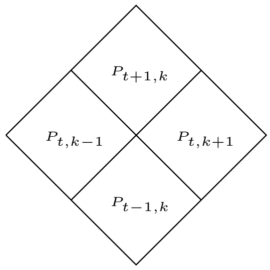
$$V^{-1}_{t+1,k-1}U^{-1}_{t+1,k-1} = f(P_{t,k-1})V^{-1}_{t,k-1}U^{-1}_{t,k-1}$$

respectively. Hence we get

$$V^{-1}_{t+1,k+1}U^{-1}_{t+1,k+1}V_{t+1,k}U^{-1}_{t+1,k} = f(P_{t,k+1})V^{-1}_{t,k+1}U^{-1}_{t,k+1}V_{t,k}U^{-1}_{t,k}f(P_{t,k-1})^{-1}$$

$\Leftrightarrow$

$$P_{t+1,k} = f(P_{t,k+1})P_{t-1,k}f(P_{t,k-1})^{-1} \tag{3.3.40}$$




which is modulo a redifinition along diagonals a quantum version of the doubly discrete sine-Gordon equation appearing in the theory of K- surfaces (1.2.13). Now $I^P_{2T} = \{(P_{2T-1,2k})_{k\in\{0..p-1\}}, (P_{2T,2k+1})_{k\in\{0..p-1\}}\}$ or $I^P_{2T-1} = \{(P_{2T+1,2k})_{k\in\{0..p-1\}}, (P_{2T,2k+1})_{k\in\{0..p-1\}}\}$, respectively, shortly denoted by $I^P_T$, is an initial configuration of arbitrary unitary periodic (face) operators, i.e. $P_{t,2p+k} = P_{t,k} \in \mathcal{A}(P_T)$, $k \in \mathbb{Z}$ which obey the commutation rules:

$$[P_{t,k}, P_{t,j}] = 0$$

$$[P_{2T\pm 1,k}, P_{2T,k+j}] = 0 \qquad \text{for } j \in \{2...2p-2\} \quad mod\, 2p$$

$$P_{t,k}P_{\tilde{t},k+1} = q^{-m}P_{\tilde{t},k+1}P_{t,k} \tag{3.3.41}$$

$t, \tilde{t} \in \{2T, 2T \pm 1\}$,$|t-\tilde{t}| = 1$ and where $\mathcal{A}(P_T)$ is the algebra of Laurent polynomials in the generators of $I^P_T$.
Consider the automorphism $\mathbf{E}_{t,k+1}\mathbf{E}_{t,k-1} : \mathcal{A}(P_T) \to \mathcal{A}(P_T)$ which is recursively defined by:

$$\begin{aligned} & \mathbf{E}_{t,k+1}\mathbf{E}_{t,k-1}(P_{t-1,k}) = \phi_{t,k+1}\phi_{t,k-1}(P_{t-1,k}) \qquad (3.3.42) \\ := \quad & R(P_{t,k+1})R(P_{t,k-1})P_{t-1,k}R(P_{t,k-1})^{-1}R(P_{t,k+1})^{-1}e^{i\xi(P^B_{t,k-1})}e^{-i\xi(P^B_{t,k+1})} \end{aligned}$$

and $\mathbf{E}_{t,k+1}\mathbf{E}_{t,k-1}$ acting trivially on all other faces along the corresponding Cauchy zig zag $\mathcal{C}_t$. Since

$$P_{t+1,k} := \mathbf{E}_{t,k+1}\mathbf{E}_{t,k-1}(P_{t-1,k}).$$

it follows by the definition (3.3.43) that :

$$\begin{aligned} & R(P_{t+1,k}) \\ = \quad & R(R(P_{t,k+1})R(P_{t,k-1})P_{t-1,k}R(P_{t,k+1})^{-1}R(P_{t,k+1})^{-1}e^{i\xi(P^B_{t,k-1})}e^{-i\xi(P^B_{t,k+1})}) \\ = \quad & R(P_{t,k+1})R(P_{t,k-1})R(e^{i\xi(P^B_{t,k-1})}e^{-i\xi(P^B_{t,k+1})}P_{t-1,k})R(P_{t,k-1})^{-1}R(P_{t,k+1})^{-1}. \end{aligned} \tag{3.3.43}$$

Define the operators ($t \in \mathbb{Z}$)

$$\mathbf{R}_{2t-1} := \prod_{k=0}^{p-1} R(P_{2t-1,2k}) \qquad \mathbf{R}_{2t} := \prod_{k=0}^{p-1} R(P_{2t,2k+1}).$$

**Proposition 3.3.10** *The operators $\mathbf{R}_t \in \mathcal{A}(P)$ defined as above evolve as:*

$$\begin{aligned}
\mathbf{R}_{2t+1} &= \mathbf{R}_{2t} \prod_{k=0}^{p-1} R(e^{i\xi(P^B_{2t,2k-1})} e^{-i\xi(P^B_{2t,2k+1})} P_{2t-1,2k}) \mathbf{R}_{2t}^{-1} \\
\mathbf{R}_{2t} &= \mathbf{R}_{2t-1} \prod_{k=0}^{p-1} R(e^{i\xi(P^B_{2t-1,2k})} e^{-i\xi(P^B_{2t-1,2k+2})} P_{2t-2,2k+1}) \mathbf{R}_{2t-1}^{-1}
\end{aligned}$$

**Proof:** Using the commutation relations in (3.3.41) and (3.3.43) and the periodicity of the face operators the proof is straightforward. □

**Corollary 3.3.11** *If $e^{i\xi(P^B_{t,k-1})} e^{-i\xi(P^B_{t,k+1})} = \mathbb{1}$ f.a. $t, x \in \mathbb{Z}$ ($k - t$ even) then $\mathbf{R}_{t+1}\mathbf{R}_t = \mathbf{R}_t\mathbf{R}_{t-1}$ is constant.*

Since in this case the evolution for the face variables is given by conjugation

$$P_{t+1,k} = \mathbf{R}_t \mathbf{R}_{t-1} P_{t-1,k} \mathbf{R}_{t-1}^{-1} \mathbf{R}_t^{-1}$$

the operator $\mathbf{R}_t\mathbf{R}_{t-1}$ can be viewed as the discrete analog of a continous hamiltonian time evolution, i.e.

$$\mathbf{R}_t \mathbf{R}_{t-1} \leftrightarrow e^{iH\Delta t_0} \qquad \Delta t_0 \text{ fixed}.$$

This should justify, why we called the automorphism constructed in (3.3.11) *almost* hamiltonian.

## 3.4 Connection to fermionic theories

### 3.4.1 Introduction of the model

Consider the previously introduced algebra of edge operators generated by the operators $\{(U_{t,k})_{k\in\{0,\dots 2p\}}, (V_{t,k})_{k\in\{0,\dots 2p\}}\}$ ($t \in \mathbb{Z}$ fixed) associated to the edges of a Cauchy zigzag $\mathcal{C}_t$, obeying the commutation relations given in 5.1.11 for the case b=0, c=0:

$$\begin{array}{lcll}
U_{t,k}U_{t,k+2l} &=& q^{-2a}U_{t,k+2l}U_{t,k} & l = 1 \dots p-1 \\
U_{t,k}U_{t,k+2m-1} &=& q^{+2a}U_{t,k+2m-1}U_{t,k} & m = 1 \dots p \\
V_{t,k}V_{t,k+n} &=& q^{+2a}V_{t,k+2l}V_{t,k} & n = 1 \dots 2p-1 \\
U_{t,k}V_{t,k+j} &=& V_{t,k+j}U_{t,k} & j \in \mathbb{Z}\backslash\{0\} \quad mod\, 2p \\
U_{t,k}V_{t,k} &=& q^{-2a}V_{t,k}U_{t,k} & \\
U_{t,k}M_{t,u} &=& q^{(-1)^{k+1}4a}M_{t,u}U_{t,k} & U_{t,k}M_{t,v} = M_{t,v}U_{t,k} \\
V_{t,k}M_{t,v} &=& q^{4a}M_{t,v}V_{t,k} & V_{t,k}M_{t,u} = M_{t,u}V_{t,k}
\end{array} \qquad (3.4.44)$$

where

$$\begin{array}{rcl} M_{t,u} & = & U_{t,2k+2p}U^{-1}_{t,2k} \\ M^{-1}_{t,u} & = & U_{t,2k+2p+1}U^{-1}_{t,2k+1} \\ M_{t,v} & = & V_{t,k+2p}V^{-1}_{t,k}. \end{array}$$

Let $\mathcal{H}^{2p}_t = \bigotimes_{k=0}^{2p-1} \mathcal{H}_{t,k}$ be a tensorproduct of equal Hilbert spaces $\mathcal{H}_{t,k}$ associated to the edges of the light cone lattice. Define $Op_k \in \mathcal{U}(\mathcal{H}^{2p})$ by

$$Op_k = \mathbb{1} \otimes \mathbb{1} \ldots \underbrace{\otimes Op \otimes}_{k'thsite} \ldots \mathbb{1} \otimes \mathbb{1}$$

where $Op \in \mathcal{U}(\mathcal{H})$ and $\mathbb{1}$ the identity on $\mathcal{H}$. Let $B, S \in \mathcal{U}(\mathcal{H})$ be unitary operators on $\mathcal{H}$ with $BS = q^{-2a}SB$.

**Proposition 3.4.1** *We fix a time $t = odd$ . Using the above definitions the operators $\{(U_{t,k})_{k\in\{0,\ldots 2p-1\}}, (V_{t,k})_{k\in\{0,\ldots 2p-1\}}, M_{t,u}, M_{t,v}\}$ may be represented in the following way:*

$$\begin{array}{ll} U_{t,2k} = B_{2k}\prod_{l=0}^{2k-1} S_l^{(-1)^l} & V_{t,2k} = S_{2k}\prod_{l=0}^{2k-1} B_l \\ U_{t,2k+1} = B_{2k+1}S_{2k+1}\prod_{l=0}^{2k} S_l^{(-1)^{l+1}} & V_{t,2k+1} = B_{2k+1}S_{2k+1}\prod_{l=0}^{2k} B_l \\ M_{t,u} = \prod_{l=0}^{2p-1} S_l^{2(-1)^l} & M_{t,v} = \prod_{l=0}^{2p-1} B_l^2 \end{array} \quad (3.4.45)$$

where $U_{t,0} = B_0$ and $V_{t,0} = S_0 = \mathbb{1} \otimes \ldots \otimes S$ and the definition should be understood modulo phase factors. If we fix a representation of $B$ and $S$ on the "small" Hilbertspace $\mathcal{H}_{t,k} = \mathcal{H}$ , assigned to the edges of the light cone lattice, then this induces a representation on the corresponding "big" Hilbertspace $\mathcal{H}^{2p}_t = \bigotimes_{k=0}^{2p-1} \mathcal{H}_{t,k}$. We notice that also if we choose $B$ and $S$ as irreducible representations on the Hilbert space $\mathcal{H}$ then the above representation will be reducible. This can be seen by Schur's Lemma and the fact that the periodic Casimirs $O_{t,2k-1} = V^{-1}_{t,2k}V_{t,2k-1}, U_{t,2k}U_{t,2k-1}$ are not multiples of the identity operator on $\mathcal{H}^{2p}_t$.

In the preceding chapters it was shown that the construction of an evolution automorphism for the above edge algebra was dependent on the chosen representation. In particular an explicit construction of such an evolution automorphism may be sometimes only possible for a subalgebra of "paired edge variables". An example of such a subalgebra is the subalgebra generated by $\{(U^2_{t,k})_{k\in\{0,\ldots 2p\}}, (V^2_{t,k})_{k\in\{0,\ldots 2p\}}\}$ and periodic Casimirs $\{O_{t,k-1} = V^{-1}_{t,k}V_{t,k-1}, U_{t,k}U_{t,k-1}\}_{k\in\{0,\ldots 2p-1\}}$. Define

$$X_{t,k} := V^2_{t,k} \qquad W_{t,k} := U^2_{t,k} \qquad M_{t,X} = M^2_{t,V}q^{-4a+2c} \qquad M_{t,W} = M^2_{t,U}q^{4a-2c}.$$

The evolution within this algebra was given by the evolution equations in 3.3.27. If we set $k_{t,k-1} = \exp\left(-\omega^2_{k+t} + \omega^1_{k-t-1}\right) = \text{constant}$, they read as ($k - t$ even):

$$M_{t,u} = const, \quad M_{t,v} = const, \quad O_{t,k-1} = const \quad (3.4.46)$$

$$X_{t+1,k-1}^{-1}X_{t,k-1} = X_{t+1,k}^{-1}X_{t,k} = \frac{k + X_{t,k}^{-1}X_{t,k-1}O_{t,k-1}^{-1}}{1 + kX_{t,k}^{-1}X_{t,k-1}O_{t,k-1}^{-1}} \tag{3.4.47}$$

$$W_{t+1,k}W_{t,k}^{-1} = W_{t+1,k-1}^{-1}W_{t,k-1} = \frac{k + W_{t,k-1}^{-1}W_{t,k}^{-1}O_{t,k-1}}{1 + kW_{t,k-1}^{-1}W_{t,k}^{-1}O_{t,k-1}} \tag{3.4.48}$$

(remember $b = 0, c = 0$) The corresponding face operators were defined by $P_{t,k-1} = X_{t,k}X_{t,k-1}O_{t,k-1}^{-1}q^{-2a} = W_{t,k-1}^{-1}W_{t,k}^{-1}O_{t,k-1}q^{-2a}$. Define the following new variables ($k - t$ even)

$$Q_{t,k-1}^{+} := W_{t,k-1}^{-1}W_{t,k}^{-1} \qquad Q_{t,k-1}^{-} := X_{t,k}^{-1}X_{t,k-1}. \tag{3.4.49}$$

Since the operators $O_{t,k-1}$ where casimirs in the edgealgebra it follows that the operators $Q_{t,k-1}^{\pm}$ are also face operators, which commute in the same way as the face operators $P_{t,k-1}$ at initial time $t$.

We define now an evolution for the variables $W_{t,k}$, $X_{t,k}$ seperately. This can be done by e.g. replacing the evolution operator $R(P_{t,k-1})$ in 3.3.11 with the operator $R(q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1})$ or $R(q^{4a}X_{t,k}^{-1}X_{t,k-1})$, respectively. Hence we define recursively (k-t even):

$$\begin{aligned}
W_{t+1,k} &= R(q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1})W_{t,k}R(q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1})e^{2i\eta(W_{t,k-1}^{-1}W_{t,k}^{-1})} \\
W_{t+1,k-1} &= R(q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1})W_{t,k-1}R(q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1})^{-1}e^{-2i\eta(W_{t,k-1}^{-1}W_{t,k}^{-1})} \\
X_{t+1,k} &= R(q^{4a}X_{t,k}^{-1}X_{t,k-1})X_{t,k}R(q^{4a}X_{t,k}^{-1}X_{t,k-1})^{-1}e^{-2i\eta(X_{t,k}^{-1}X_{t,k-1})} \\
X_{t+1,k-1} &= R(q^{4a}X_{t,k}^{-1}X_{t,k-1})X_{t,k-1}R(q^{4a}X_{t,k}^{-1}X_{t,k-1})^{-1}e^{-2i\eta(X_{t,k}^{-1}X_{t,k-1})}
\end{aligned} \tag{3.4.50}$$

$2\eta = \xi$ as before in (3.3.5). We obtain the evolution equations

$$\begin{aligned}
W_{t+1,k}W_{t,k}^{-1} = W_{t+1,k-1}^{-1}W_{t,k-1} &= \frac{R(q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1})}{R(q^{-4a}W_{t,k-1}^{-1}W_{t,k}^{-1})}e^{-2i\eta(W_{t,k-1}^{-1}W_{t,k}^{-1})} \\
X_{t+1,k-1}^{-1}X_{t,k-1} = X_{t+1,k}^{-1}X_{t,k} &= \frac{R(q^{4a}X_{t,k}^{-1}X_{t,k-1})}{R(q^{-4a}X_{t,k}^{-1}X_{t,k-1})}e^{-2i\eta(X_{t,k}^{-1}X_{t,k-1})}
\end{aligned}$$

Such that if $q^{4a}X_{t,k}^{-1}X_{t,k-1}$,$q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1}$ satisfy the preliminaries of theorem 3.3.5 with $q^{-8a} = q^m$ this gives us the equations:

$$\begin{aligned}
W_{t+1,k}W_{t,k}^{-1} = W_{t+1,k-1}^{-1}W_{t,k-1} &= \frac{k + q^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1}}{1 + kq^{4a}W_{t,k-1}^{-1}W_{t,k}^{-1}} \\
X_{t+1,k-1}^{-1}X_{t,k-1} = X_{t+1,k}^{-1}X_{t,k} &= \frac{k + q^{4a}X_{t,k}^{-1}X_{t,k-1}}{1 + kq^{4a}X_{t,k}^{-1}X_{t,k-1}}
\end{aligned} \tag{3.4.51}$$

Using (3.4.51) we obtain the following two sine-Gordon type equations

$$Q_{t+1,k}^{\pm} = \frac{1 + q^{-4a}kQ_{t,k-1}^{\pm}}{k + q^{-4a}Q_{t,k-1}^{\pm}}\frac{k + q^{4a}Q_{t,k+1}^{\pm}}{1 + kq^{4a}Q_{t,k+1}^{\pm}}Q_{t-1,k}^{\pm}. \tag{3.4.52}$$

Note that $Q_{t,k-1}^{+}Q_{t,k-1}^{-} = O_{t,k-1}$.

### 3.4.2 Light cone shifts

The doubly discrete sine-Gordon equation, as well as the above described equations of sine-Gordon type are, as in the continous case, invariant under light cone shifts, i.e. if $g_{t,k} : \mathbb{Z}^2 \to \mathbb{R}$ is a solution to the classical vertex equation then $g_{t\pm n,k\pm n}$ is also a solution. In this sense space time shifts can be lifted to automorphisms on covariant phase space and can be interpreted as symplectomorphisms [16].

In the previous section we found a quantization of another (yet trivial) symplectomorphism ([45, 26]) on phase space, namely time evolution. It would be now only consequent to find quantized analogs of the above mentioned light cone shifts. This will be done by defining quantized space translations of half the lattice spacing distance and then applying the time automorphism. Since translations of half the lattice spacing distance are hard to define on the vertex operators, as one would have to go over to the dual lattice, one has to restrict oneself to the edge algebra $\mathcal{A}(X_T)$.

For constructing the above mentioned space shifts, we will follow an idea developped in [21], where such space shifts were suggested for the case of a special choice of vertex monodromies. As it will turn out the treatment of the more general case will result in a possibility to fix the above roots in a very natural way. Let again $m = 8a$, $2\eta = \xi$.

**Lemma 3.4.2** *The quotient of the two vertex monodromies*

$$\begin{aligned} M^{(1)}(M^{(2)})^{-1} &= P_{2t\pm1,0}P_{2t\pm1,2}\dots P_{2t\pm1,2p-2}P^{-1}_{2t,2p-1}P^{-1}_{2t,2p-3}\dots P^{-1}_{2t,1} \\ &= q^{-2mp+m}X^2_{2t,0}X^{-2}_{2t,1}X^2_{2t,2}\dots X^2_{2t,2p-1}M_X \\ &= q^{-2mp+m}X^2_{2t+1,0}X^{-2}_{2t+1,1}X^2_{2t+1,2}\dots X^2_{2t+1,2p-1}M_X \end{aligned}$$

*is a Casimir in $\mathcal{A}(X_T)$.*

**Demand 3.4.3** *For establishing quantum space time shifts demand*

a.) *The Casimirs*

$$P_{2t-1,0}P_{2t-1,2}\dots P_{2t-1,2p-2}P^{-1}_{2t,2p-1}P^{-1}_{2t,2p-3}\dots P^{-1}_{2t,1}$$

*and $P^B_{t,k-1}$ (B as before) shall be multiples of the identity within $\mathcal{A}(X_T)$.*

b.) *The roots $e^{i\xi_0(P^B_{t,k-1})} = (P_{t,k-1}(-1)^{B-1})^{-\frac{1}{B}}$ (see 3.3.17) shall be fixed in such a way that*

$$\prod_{k=0}^{p-1}(P^B_{2t-1,2k}(-1)^{B-1})^{\frac{1}{B}}\prod_{k=0}^{p-1}(P^B_{2t,2k-1}(-1)^{B-1})^{-\frac{1}{B}} = \prod_{k=0}^{p-1}P_{2t-1,2k}\prod_{k=0}^{p-1}P^{-1}_{2t,2k-1}$$

*By the definition of an almost hamiltonian quantum evolution demand a.) holds automatically for all times $t$, if it is true at an initial time $T$. The same is valid for demand b.) which will become evident soon, therefore the index $t$ in the above demands refers to all times.*

Define

$$S_t^{-1} := \prod_{k=2}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1}) = R_0(X_{t,2p}^{-1}X_{t,2p-1})R_0(X_{t,2p-1}^{-1}X_{t,2p-2})\dots R_0(X_{t,2}^{-1}X_{t,1}) \tag{3.4.53}$$

**Proposition 3.4.4** *For all $k \in \mathbb{Z}$*

$$S_t^{-1}X_{t,k}S_t = q^{-m}e^{i\xi_0((X_{t,k}^{-1}X_{t,k-1})^B)}X_{t,k-1}$$

*where $e^{i\xi_0(x^B)} = (x^B(-1)^{B-1})^{-\frac{1}{B}}$ as in (3.3.17).*

**Proof:**
If $n \in \{2\dots 2p\}$ then by the commutation rules of the edge variables

$$\begin{aligned}
&S_t^{-1}X_{t,n}S_t = \\
= &\prod_{k=n}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,n}\prod_{k=n}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1} \\
= &\prod_{k=n+1}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})\frac{R_0(X_{t,n}^{-1}X_{t,n-1})}{R_0(q^mX_{t,n}^{-1}X_{t,n-1})}X_{t,n}\prod_{k=n+1}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1} \\
= &\prod_{k=n+1}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,n}^{-1}X_{t,n-1}X_{t,n}\prod_{k=n+1}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)} \\
= &\ q^{-m}\prod_{k=n+1}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,n-1}\prod_{k=n+1}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)} \\
= &\ q^{-m}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}X_{t,n-1}
\end{aligned}$$

Analogously one obtains

$$\begin{aligned}
&S_t^{-1}X_{t,1}S_t = \\
= &\prod_{k=3}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,2}^{-1}X_{t,1}^2\prod_{k=3}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1}e^{i\xi_0((X_{t,2}^{-1}X_{t,1})^B)} \\
= &\prod_{k=4}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})\frac{R_0(X_{t,3}^{-1}X_{t,2})}{R_0(q^{-m}X_{t,3}^{-1}X_{t,2})}X_{t,2}^{-1}X_{t,1}^2\prod_{k=4}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})e^{i\xi_0((X_{t,2}^{-1}X_{t,1})^B)}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{k=4}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})q^{2m}X_{t,3}X_{t,2}^{-2}X_{t,1}^2 \prod_{k=4}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})e^{-i\xi_0((X_{t,3}^{-1}X_{t,2})^B)+i\xi_0((X_{t,2}^{-1}X_{t,1})^B)} \\
&= q^{2m(p-1)}X_{t,2p}^{-1}\prod_{k=1}^{2p-1} X_{t,k}^{2(-1)^{k+1}} \prod_{k=1}^{2p} e^{(-1)^k i\xi_0((X_{t,k}^{-1}X_{t,k-1})^B)}\, e^{i\xi_0((X_{t,1}^{-1}X_{t,0})^B)} \\
&= q^{-m}e^{i\xi_0((X_{t,1}^{-1}X_{t,0})^B)}X_{t,0} \qquad (3.4.54)
\end{aligned}
$$

if we suppose that demand b.) holds. With $S_t^{-1}M_XS_t = M_X$ the assertion follows.
□
It is easy to show that:

$$(S_t^{(l)})^{-1}X_{t,n}S_t^{(l)} = q^{-m}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}X_{t,n-1}$$

where

$$(S_t^{(l)})^{-1} := \prod_{k=2+l}^{2p+l\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})$$

Therefore the index $l$ is irrelevant and will be skipped, also if $(S_t^{(l)})^{-1}$ instead of $(S_t^{(0)})^{-1}$ is used. Clearly one can also define an automorphism of the above kind by redefining

$$\hat{S}_t^{-1} := \prod_{k=2+l}^{2p+l\leftarrow} R_0(\alpha X_{t,k}^{-1}X_{t,k-1}) \qquad \alpha \in S^1,$$

which will act as:

$$\hat{S}_t^{-1}X_{t,n}\hat{S}_t = \alpha q^{-m}e^{-i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}X_{t,n-1},$$

a fact to be used later.
The automorphism $\mathbf{S}_t^{-1} : \mathcal{A}(X_T) \to \mathcal{A}(X_T)$ defined on the operators $X_{t,n}$ as

$$\mathbf{S}_t^{-1}(X_{t,n}) := q^m e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}S_t^{-1}X_{t,n}S_t = X_{t,n-1}$$

can be interpreted as a shift of these edge operators in space direction. As an automorphism on $\mathcal{A}(P_T)$, the picture of the action of $\mathbf{S}_t^{-1}$ is a little different, we find

$$
\begin{aligned}
S_t^{-1}P_{t,k-1}S_t &= P_{t-1,k-2}\, e^{-i\xi_0(P_{t,k-1}^B)} + e^{i\xi_0(P_{t-1,k-2}^B)} \\
S_t^{-1}P_{t-1,k-2}S_t &= P_{t,k-3}\, e^{-i\xi_0(P_{t,k-2}^B)} + e^{i\xi_0(P_{t-1,k-3}^B)}
\end{aligned}
$$

Hence $\mathbf{S}_t$ applied to face operators is rather an up and down shift in lightcone direction than a shift in space:

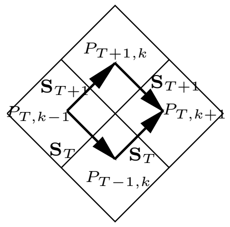



Figure 2.1

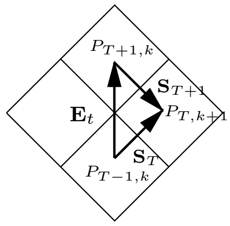



Figure 2.2

Fix an initial time e.g. $T=$ odd. Then by definition the operators

$$\begin{aligned} S_T^{-1} &= R_0(P_{T-1,2p-1})R_0(P_{T,2p-2})\dots R_0(P_{T-1,1}) \\ S_{T+1}^{-1} &= R_0(P_{T+1,2p-1})R_0(P_{T,2p-2})\dots R_0(P_{T+1,1}) \end{aligned}$$

are given by the choice of initial operators $\{(P_{T-1,2k+1})_{k\in\{0\dots p-1\}}, (P_{T,2k})_{k\in\{0\dots p-1\}}\}$ and the time evolved face operators $(P_{T+1,2k+1})_{k\in\{0\dots p-1\}}$. The operator $P_{T+1,2k+1}$ can now be obtained by shifting the operator $P_{T,2k}$ or by applying the time evolution to $P_{T-1,2k+1}$ , i.e. the in figure 2.2 depictured shifts have to commute.

By the commutation relations of the face operators it is straightforward to find:

**Lemma 3.4.5**

$$\begin{aligned} &R_0(P_{T+1,2p-1})R_0(P_{T,2p-2})\dots R_0(P_{T+1,1}) \\ &= \mathbf{R}_T R_0(e^{-i\xi_0(P^B_{T,2p})+i\xi_0(P^B_{T,2p-2})}P_{T-1,2p-1})R_0(P_{T,2p-2}) \\ &\dots R_0(e^{-i\xi_0(P^B_{T,2})+i\xi_0(P^B_{T,0})}P_{T-1,1})\mathbf{R}_T^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} & S_{T+1}^{-1}P_{T,2k}S_{T+1} \stackrel{!}{=} P_{T+1,2k-1}e^{-i\xi_0(P^B_{T,2k})+i\xi_0(P^B_{T+1,2k-1})} \\ \stackrel{3.4.5}{=} \; & \mathbf{R}_T R_0(e^{-i\xi_0(P^B_{T,2p})+i\xi_0(P^B_{T,2p-2})}P_{T-1,2p-1})R_0(P_{T,2p-2}) \\ & \dots R_0(e^{-i\xi_0(P^B_{T,2})+i\xi_0(P^B_{T,0})}P_{T-1,1})P_{T,2k} \\ & (\mathbf{R}_T R_0(e^{-i\xi_0(P^B_{T,2p})+i\xi_0(P^B_{T,2p-2})}P_{T-1,2p-1})R_0(P_{T,2p-2}) \\ & \dots R_0(e^{-i\xi_0(P^B_{T,2})+i\xi_0(P^B_{T,0})}P_{T-1,1}))^{-1} \\ = \; & \mathbf{R}_T P_{T-1,2k-1}\mathbf{R}_{T-1}^{-1}e^{-i\xi_0(P^B_{T,2k})+i\xi_0((e^{-i\xi_k(P^B_{T,2k})+i\xi_k(P^B_{T,2k-2})}P_{T-1,2k-1})^B)} \\ & e^{-i\xi_k(P^B_{T,2k})+i\xi_k(P^B_{T,2k-2})} \\ = \; & P_{T+1,2k-1}e^{-i\xi_0(P^B_{T,2k})+i\xi_0((e^{-i\xi_k(P^B_{T,2k})+i\xi_k(P^B_{T,2k-2})}P_{T-1,2k-1})^B)} \end{aligned}$$

Comparing both sides of the equation, one obtains finally the compatibility condition:

$$e^{i\xi_0(P^B_{T+1,2k-1})} \stackrel{!}{=} e^{-i\xi_k(P^B_{T,2k})+i\xi_k(P^B_{T,2k-2})}e^{i\xi_0(P^B_{T-1,2k-1})} \tag{3.4.55}$$

For defining the evolution we had already fixed the roots on the right hand side of equation (3.4.55), so that equation (3.4.55) determines the roots at one time step further.

Equation (3.4.55) fixes the roots at $k = 0$. It is easy to see that one can stay on one leave, when extending to arbitrary $k \in [0, 1)$. Hence also the evolution of the roots $e^{-i\xi_k(P)}$ has been now defined. As a direct consequence of equation (3.4.55), it follows that demand b.) holds for all times since the classical monodromies are as well as their quantum counterparts integrals of motion [38].

Note that if one takes the $B'th$ power of the above equation then one obtains

$$P^B_{T+1,2k-1} = \frac{k^B + P^B_{T,2k}(-1)^{B-1}}{1 + k^B P^B_{T,2k}(-1)^{B-1}} \frac{1 + k^B P^B_{T,2k-2}(-1)^{B-1}}{k^B + P^B_{T,2k-2}(-1)^{B-1}} P^B_{T-1,2k-1}.$$

Hence the "classical" variables $P^B_{t,k}$ satisfy also an equation of sine-Gordon type. A fact which was first noticed in [52] by using the commutation relations of the face operators and computing $P^B_{T+1,2k-1}$.

It is now straightforward to show that as an immediate consequence

$$\mathbf{S}_{T+1}(X_{T+1,k}) = \mathbf{E}_T \circ \mathbf{S}_T \circ \mathbf{E}_T^{-1}(X_{T+1,k})$$

i.e. shifts in time and space direction commute if (3.4.55) is satisfied.

The picture of the above developped quantum evolution looks considerably complicate. It simplifies by a great amount, if one restricts oneself to the case of corollary (3.3.11):

**Proposition 3.4.6** *Let $\mathbf{S}_T$ be an automorphism on $\mathcal{A}(X_T)$ such that at an initial time $T$*

$$\mathbf{R}_T = \mathbf{S}_T(\mathbf{R}_{T-1}) = \mathbf{S}_T^{-1}(\mathbf{R}_{T-1}) \tag{3.4.56}$$

*where recursively*

$$\mathbf{R}_{t+1} = \mathbf{R}_t \mathbf{R}_{t-1} \mathbf{R}_t^{-1} \tag{3.4.57}$$

*then $\mathbf{R}_t = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{t-1})$ for all $t \in \mathbb{Z} > T$.*

**Proof:**
By (3.4.56) and (3.4.57)

$$\mathbf{R}_T = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{T-1})\mathbf{R}_T^{-1} \tag{3.4.58}$$
$$\mathbf{R}_{T+1} = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_T)\mathbf{R}_T^{-1} \tag{3.4.59}$$

For completing the inductional argument assume that the following is true

$$\mathbf{R}_{T+n} = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{T-1+n})\mathbf{R}_T^{-1} \tag{3.4.60}$$
$$\mathbf{R}_{T+1+n} = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{T+n})\mathbf{R}_T^{-1} \tag{3.4.61}$$

Hence

$$\begin{aligned}
\mathbf{R}_{T+n+2} &\overset{(3.4.57)}{=} \mathbf{R}_{T+n+1}\mathbf{R}_{T+n}\mathbf{R}_{T+n+1}^{-1} \\
&\overset{(3.4.60,3.4.61)}{=} \mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n})\mathbf{R}_T^{-1}\mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T-1+n})\mathbf{R}_T^{-1}(\mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n})\mathbf{R}_T^{-1})^{-1} \\
&\overset{(3.4.57)}{=} \mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n+1})\mathbf{R}_T^{-1} \qquad (3.4.62)
\end{aligned}$$

analogously $\mathbf{R}_{T+n+3} \overset{(3.4.61,3.4.62)}{=} \mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n+2})\mathbf{R}_T^{-1}$.
□

**Proposition 3.4.7** *If $\mathbf{S}_T$ is an automorphism such at initial time $T$*

$$X_{T,k} = \mathbf{S}_T(X_{T,k-1}) = \mathbf{S}_T^{-1}(X_{T,k+1}) \qquad \text{and} \quad X_{t+1,k} = \mathbf{R}_t X_{t,k}\mathbf{R}_t^{-1}$$

*for all $t \in \mathbb{N}$ and with $\mathbf{R}_t$ as in proposition (3.4.6) then*

$$X_{t+1,k} = \mathbf{R}_T\mathbf{S}_T(X_{t,k-1})\mathbf{R}_T^{-1}$$

**Proof:**
By induction as above and by the use of proposition 3.4.6.
□
The connection to models of statistical mechanics is now evident. We find

$$\begin{aligned}
X_{t+2,k+1} &= \mathbf{R}_T\mathbf{S}_T^{-1}(\mathbf{R}_T\mathbf{S}_T(X_{t,k-1})\mathbf{R}_T^{-1})\mathbf{R}_T^{-1} \\
&= \mathbf{R}_T\mathbf{S}_T^{-1}(\mathbf{R}_T)X_{t,k+1}\mathbf{S}_T^{-1}(\mathbf{R}_T^{-1})\mathbf{R}_T^{-1}. \qquad (3.4.63)
\end{aligned}$$

Moreover $\mathbf{R}_T$ is a product of "local amplitudes" $R(P_{T,k-1})$ associated to the faces at time $T$ within the light cone lattice, hence $\mathbf{S}_T^{-1}(\mathbf{R}_T)$ is a product of "local amplitudes" $R(P_{t,k-1})$ associated to faces which are shifted in lightcone direction of the original faces. Because of (3.4.63) this picture is the same all over the lattice, hence we can interpret $R_T$ as a kind of transfermatrix (though with complex weights).

Another fortunate consequence of the above is that for investigating the evolution it suffices to control the first time step, everything else is obtained by applying the light cone shifts $\mathbf{E}_T \circ \mathbf{S}_T$. this is espiacially important for the construction of integrals of motion, since if one finds an operator $H_T$, which commutes with the above light cone shifts, then this will be automatically an integral of motion.

In the next section an example of such a 'static' quantum field theory will be discussed.

### 3.4.3 Relations to the massive Thirring model

We will work now with the subalgebras generated by $(W_{t,k})_{k\in\{0,\ldots 2p\}}$ and $(X_{t,k})_{k\in\{0,\ldots 2p\}}$ ($t \in \mathbb{Z}$ fixed) carrying the induced comutation relations given in (3.4.44) and the new evolution automorphism given in (3.4.50).

Define $B, S, F, A \in U(2,\mathbb{C}) = \mathcal{U}(\mathbb{C}^2)$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.4.64}$$
$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad A = \frac{1}{1-i}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \qquad C = \frac{1-i}{\sqrt{2}}AF = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

The notation is adopted from viewing $B$ as a "**B**oost" and $S$ as a "**S**hift" operator. $F$ is a discrete **F**ouriertransform with

$$F = F^{-1} \qquad FBF = S.$$

The operator $A$ diagonalizes $BS$ moreover we collect the following identities:

$$\begin{array}{rclrcl} A^{-1}BA & = & S & CBC^{-1} & = & B \\ ABA^{-1} & = & -iBS & CSC^{-1} & = & -iBS \\ A^{-1}SA & = & -iBS & CBSC^{-1} & = & -iS. \end{array} \tag{3.4.65}$$

$B, S, F, A$ are operators acting on small Hilbertspace $\mathcal{H} = \mathbb{C}^2$. The Hilbertspaces $\mathcal{H}$ assigned to each edge are still assumed to be equal. Since we are running out of letters we will abuse our notation and define $A$ and $F$ also as operators on $\mathcal{H}^{2p} = \bigotimes_{k=0}^{2p-1}\mathbb{C}^2$, *i.e.*

$$A_k = \mathbb{1} \otimes \mathbb{1} \ldots A \ldots \mathbb{1} \otimes \mathbb{1} \qquad F_k = \mathbb{1} \otimes \mathbb{1} \ldots F \ldots \mathbb{1} \otimes \mathbb{1}$$
$$A = \bigotimes_{k=0}^{2p-1} A_k \qquad F = \bigotimes_{k=0}^{2p-1} F_k$$

It will be evident from the context, which definition is meant.

After fixing phase factors in a desired way, the by (3.4.45) induced representation reads as:

$$\begin{array}{ll} W_{t,2k} = B_{2k}\prod_{l=0}^{2k-1} S_l & X_{t,2k} = S_{2k}\prod_{l=0}^{2k-1} B_l \\ W_{t,2k+1} = B_{2k+1}S_{2k+1}\prod_{l=0}^{2k} S_l & X_{t,2k+1} = -B_{2k+1}S_{2k+1}\prod_{l=0}^{2k} B_l \\ M_{t,X} = M_{t,W} = \mathbb{1} & \end{array}$$

where again we have fixed an initial time $t =$ odd and $B$ and $S$ are now explicitely given by 3.4.64.

We see that in the above representation the monodromies are the identity operator on $\mathcal{H}^{2p}$ and hence the edge variables are periodic. Moreover we find that at our initial time $t =$ odd :

$$FX_{t,k}F = W_{t,k} \tag{3.4.66}$$

$$W_{t,2k}^2 = X_{t,2k}^2 = \mathbb{1} \qquad W_{t,2k+1}^2 = X_{t,2k+1}^2 = -\mathbb{1}\,. \tag{3.4.67}$$

With

$$Q^-_{t,k} = X^{-1}_{t,k+1} X_{t,k} \qquad Q^+_{t,k} = F Q^-_{t,k} F$$

follows that

$$(Q^{\pm}_{t,k})^2 = \mathbf{1}\,. \tag{3.4.68}$$

**Corollary 3.4.8** *from theorem 3.3.5. The operator* $R(iQ^{\pm}) \in U(\mathbb{C}^2 \otimes \mathbb{C}^2)$ *given by*

$$R(iQ^{\pm}) = \frac{1}{\sqrt{2(1+k^2)}}(k + iQ^{\pm})(1 - iQ^{\pm}) = \frac{1}{\sqrt{2(1+k^2)}}(1 - kiQ^{\pm})(1 + iQ^{\pm})$$

*with* $k \in \mathbb{R}$ *and* $(Q^{\pm})^2 = \mathbb{1}$ *satisfies the functional equation*

$$\frac{R(iQ^{\pm})}{R(-iQ^{\pm})} = \frac{k + iQ^{\pm}}{1 + kiQ^{\pm}}$$

**Proof:**
The right input for Theorem (3.3.5) is $q = \exp(i\pi), m = 1, N = 2$. It follows imediately that $\exp(i\xi)$ can be chosen to be $\exp(i\xi) = 1$, since

$$e^{2i\xi} = \frac{k^2 + (iQ^{\pm})^2(-1)}{1 + k^2(iQ^{\pm})^2(-1)} = 1$$

Choosing $e^{i\xi} = 1$ we obtain as an immediate solution

$$\tilde{R}(iQ^{\pm}) = 1 + \frac{1-k}{1+k}(iQ^{\pm})$$

But

$$\tilde{R}(iQ^{\pm})\overline{\tilde{R}(iQ^{\pm})} = (1 + (\frac{1-k}{1+k})^2)\mathbb{1}$$

hence

$$R(iQ^{\pm}) = \frac{1}{\sqrt{1 + (\frac{1-k}{1+k})^2}}(1 + \frac{1-k}{1+k}(iQ^{\pm}))$$

is a solution, rewriting $R(iQ^{\pm})$ gives the assertion also at $k = -1$.
□
Following 3.4.50 the evolution on the face $(t, k-1)$ is now explicitely given by the recursive procedure :

$$X_{t+1,k} = R(iQ^-_{t,k-1}) X_{t,k} R(iQ^-_{t,k-1})^{-1} e^{2i\xi((Q^-_{t,k-1})^B)} \tag{3.4.69}$$

$$X_{t+1,k-1} = R(iQ^-_{t,k-1}) X_{t,k-1} R(iQ^-_{t,k-1})^{-1} e^{2i\xi((Q^-_{t,k-1})^B)} \tag{3.4.70}$$

$$W = FXF,$$

where t now arbitrary with k - t even.

Considering 3.4.68 we find

$$\frac{k+iQ^{\pm}}{1+kiQ^{\pm}}\frac{1-kiQ^{\pm}}{1-kiQ^{\pm}} = \frac{1}{1+k^2}(2k+(1-k^2)iQ^{\pm}). \tag{3.4.71}$$

Define

$$c{:=}\ \frac{2k}{1+k^2} \qquad b{:=}\ -i\frac{1-k^2}{1+k^2}.$$

$b$ and $c$ obey the unitarity conditions:

$$|b|^2+|c|^2=1 \qquad b\bar{c}+\bar{b}c=0$$

so we set

$$b =: -i\sin\phi \qquad c =: \cos\phi \tag{3.4.72}$$

Rewriting $R(iQ)$ in terms of the angle $\phi$ we find

$$R(iQ) = \cos\frac{\phi}{2}\mathbb{1} + i\sin\frac{\phi}{2}Q,$$

In terms of the constants $b$ and $c$:

$$R(iQ) = \frac{i}{\sqrt{2(1-c)}}(b\mathbb{1} + (1-c)Q)$$

Let us again fix an initial time $t$ = odd. Using the above definitions of the constants $b$, $c$ , the identities (3.4.67, 3.4.68) and (3.4.71), we get from 3.4.69 the following evolution equations for the edge operators, where we confine ourselves to the investigation of the evolution in the subalgebra generated by the edge operators $(X_{t,k})_{k\in\{0,\dots 2p\}}$, the evolution within the subalgebra generated by the edge operators $(W_{t,k})_{k\in\{0,\dots 2p\}}$ can be obtained analogously:

$$\begin{aligned} X_{t+1,2k} &= (c\mathbb{1} - bQ^-_{t,2k})X_{t,2k} = bX_{t,2k+1} + cX_{t,2k} \\ X_{t+1,2k+1} &= (c\mathbb{1} - bQ^-_{t,2k})X_{t,2k+1} = bX_{t,2k} + cX_{t,2k+1} \end{aligned} \tag{3.4.73}$$

Observe that for obtaining the evolution equations (3.4.73) we didn't use the explicit representation (3.4.64) but only the equations (3.4.67) and (3.4.68).
Now using (3.4.73) we find

$$X^2_{t+1,2k} = \mathbb{1} \qquad X^2_{t+1,2k+1} = -\mathbb{1} \tag{3.4.74}$$

and hence $Q^-_{t+1,2k-1} := -X_{t+1,2k-1}X^{-1}_{t+1,2k}$ satisfies

$$(Q^-_{t+1,2k-1})^2 = \mathbb{1} \tag{3.4.75}$$

Using (3.4.75) we obtain

$$\begin{aligned} X_{t+2,2k} &:= (c\mathbb{1} - bQ^-_{t+1,2k-1})X_{t+1,2k} = bX_{t+1,2k-1} + cX_{t+1,2k} \\ X_{t+2,2k-1} &:= (c\mathbb{1} - bQ^-_{t+1,2k-1})X_{t+1,2k-1} = bX_{t+1,2k} + cX_{t+1,2k-1} \end{aligned}$$

Hence inductively we obtain the following picture of a local evolution

$$\begin{pmatrix} X_{t+1,k} \\ X_{t+1,k+1} \end{pmatrix} = \begin{pmatrix} \cos\phi & -i\sin\phi \\ -i\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} X_{t,k} \\ X_{t,k+1} \end{pmatrix} \tag{3.4.76}$$

with

$$X^2_{t,2k} = \mathbb{1}\,, \quad X_{t,2k-1} = -\mathbb{1} \quad \text{f.a.} \quad t \in \mathbb{Z}.$$

Return to the representation given in (3.4.64). Define

$$\psi_{t,k} := \frac{1}{2}(X_{t,k-1} - iCX_{t,k-1}C^{-1}).$$

Hence for an initial time t:

$$\psi_{t,k} = \psi_k = \sigma^-_k \prod_{l=1}^{k-1} \sigma^z_l = \mathbb{1} \otimes \ldots \otimes \sigma^- \otimes \sigma^z \ldots \sigma^z$$

where

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = B$$

The operators $\psi_k$ are called Fermi operators and obey the Fermi commutation relations

$$[\psi_n, \psi^\dagger_n] = 1 \quad [\psi_n, \psi_m] = [\psi_n, \psi^\dagger_m] = [\psi^\dagger_n, \psi^\dagger_m] = 0.$$

Since the evolution equations (3.4.76) are linear the evolved Fermi operators also satisfy an evolution of the type (3.4.76). It is possible to describe the evolution of the fermionic operators in terms of lattice shifts in light cone direction. First we note that the evolution for the edge operators to the next time step is given by

$$X_{T+1,n} = \mathbf{R}_T X_{T,n} \mathbf{R}_T^{-1} \quad \text{where} \quad \mathbf{R}_T = \prod_{l=0}^{p-1} R_k(iP_{T,2l}).$$

The shift matrix shall be given by:

$$S_T^{-1} := R_0(iP_{T,2p-2})R_0(iP_{T-1,2p-3})\ldots R_0(iP_{T-1,1})R_0(iP_{2T,0})$$

which defines the following shiftautomorphism on the edge algebra:

$$\begin{aligned} \mathbf{S}_T^{-1}(X_{T,n}) &:= iS_T^{-1}X_{T,n}S_T = X_{T,n-1} \qquad n \in \{1\ldots 2p-1\} \\ \mathbf{S}_T^{-1}(X_{T,0}) &:= i(-1)^{p-1}S_T^{-1}X_{T,0}S_T = X_{T,2p-1} \end{aligned}$$

**Lemma 3.4.9** *The matrices* $\mathbf{C}P_{T,2n}\mathbf{C}^{-1}$ *and* $\mathbf{C}P_{T,2n+1}\mathbf{C}^{-1}$ $(n \in \{0 \dots p-1\})$ *commute with all generators* $I_T^X = (X_{T,n})_{n\in\{0..2p\}}$ *of* $\mathcal{A}(X_T)$.

Hence

$$\begin{aligned}
&\mathbf{R}_T\mathbf{C}\mathbf{R}_T\mathbf{C}^{-1}S_T^{-1}\mathbf{C}S_T^{-1}\mathbf{C}^{-1}(X_{T,n} - i\mathbf{C}X_{T,n}\mathbf{C}^{-1})\mathbf{C}S_T\mathbf{C}^{-1}S_T\mathbf{C}\mathbf{R}_T^{-1}\mathbf{C}^{-1}\mathbf{R}_T^{-1} \\
= \;& \mathbf{R}_T S_T^{-1}X_{T,n}S_T\mathbf{R}_T^{-1} - i\mathbf{C}\mathbf{R}_T S_T^{-1}X_{T,n}S_T\mathbf{R}_T^{-1}\mathbf{C}^{-1} \\
= \;& -i(\mathbf{R}_T\mathbf{S}_T^{-1}(X_{T,n})\mathbf{R}_T^{-1} - i\mathbf{C}\mathbf{R}_T\mathbf{S}_T^{-1}(X_{T,n})\mathbf{R}_T^{-1}\mathbf{C}^{-1})
\end{aligned}$$

$n \in \{1 \dots p-1\}$, analogous for $X_{T,0}$.
Clearly this defines lightcone shifts on the operators

$$\psi_{T,n+1} := \frac{1}{2}(X_{T,n} - i\mathbf{C}X_{T,n}\mathbf{C}^{-1}) = \sigma_n^- \prod_{l=0}^{n-1}\sigma_l^z,$$

i.e.

$$\psi_{T+1,n+1} := \tilde{\mathbf{R}}_T\tilde{\mathbf{S}}_T(X_{T,n})\tilde{\mathbf{R}}_T^{-1}$$

with $n \in \{0 \dots p-1\}$

$$\tilde{\mathbf{R}}_T := \mathbf{R}_T\mathbf{C}\mathbf{R}_T\mathbf{C}^{-1} \qquad \tilde{\mathbf{S}}_T^{-1}(\psi_{T,n}) := iS_T^{-1}\mathbf{C}S_T^{-1}\mathbf{C}^{-1}\psi_{T,n}S_T\mathbf{C}S_T\mathbf{C}^{-1}$$

and

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = B.$$

The shift matrices $S_T$ and $S_T^{-1}$, as well as the evolution matrix $\mathbf{R}_T$ are products of bilocal operators

$$\begin{aligned}
R_k(iP_{T,2n}) &= \mathbb{1} \otimes \dots R_k(iBS \otimes BS) \dots \otimes \mathbb{1} \quad n \in \{0 \dots p-1\} \\
R_k(iP_{T-1,2n+1}) &= \mathbb{1} \otimes \dots R_k(-iS \otimes S) \dots \otimes \mathbb{1} \quad n \in \{0 \dots p-2\}.
\end{aligned}$$

Hence the same holds fot $\tilde{\mathbf{R}}_T$ and $\tilde{S}_T$. A straightforward computation gives

$$\begin{aligned}
R_k(iBS \otimes BS)\mathbf{C}R_k(iBS \otimes BS)\mathbf{C}^{-1} &= R_k(iBS \otimes BS)R_k(-iS \otimes S) \\
&= \mathbb{1} \otimes \dots \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \dots \otimes \mathbb{1} = \\
= R_k(-iS \otimes S)\mathbf{C}R_k(-iS \otimes S)\mathbf{C}^{-1}
\end{aligned}$$

The shift matrices

$$\tilde{S}_T^{-1} = \tilde{R}_0(iP_{T,2p-2})\tilde{R}_0(iP_{T-1,2p-3})\tilde{R}_0(iP_{T,2p-4}) \dots \tilde{R}_0(iP_{T,0})$$

act on the fermionic operators $\psi_{T,k}$ as

$$
\begin{aligned}
& \tilde{S}_T^{-1}\psi_{T,2n}\tilde{S}_T = \\
= \;& \tilde{R}_0(iP_{T,2n-2})\psi_{T,2n}\tilde{R}_0(iP_{T,2n-2})^{-1} \\
= \;& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\sigma^-_{2n-1}\otimes\sigma^z_{2n-2}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \prod_{l=0}^{2n-3}\sigma^z_l \\
= \;& -i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\sigma^-_{2n-1}\otimes\sigma^z_{2n-2}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \prod_{l=0}^{2n-3}\sigma^z_l \\
= \;& -i\mathbf{V}^{-1}\psi_{T,2n}\mathbf{V}
\end{aligned}
\tag{3.4.77}
$$

$n \in \{0\ldots p-1\}$ with

$$
V_n = V_n^{-1} = \mathbb{1}\otimes\ldots\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{(n+1,n)'th\ site}\ldots\otimes\mathbb{1} \qquad \mathbf{V}^{-1} = V^{-1}_{2p-2}V^{-1}_{2p-3}\ldots V_0^{-1}
$$

In a similar way

$$
\begin{aligned}
\tilde{S}_T^{-1}\psi_{T,2n+1}\tilde{S}_T &= -i\mathbf{V}^{-1}\psi_{T,2n+1}\mathbf{V} \quad n\in\{1\ldots p-1\} \\
\tilde{S}_T^{-1}\psi_{T,1}\tilde{S}_T &= i(-1)^p\mathbf{V}^{-1}\psi_{T,1}\mathbf{V},
\end{aligned}
$$

so that finally

$$
\begin{aligned}
\tilde{\mathbf{S}}_T^{-1}(\psi_{T,n}) &= i\tilde{S}_T^{-1}\psi_{T,n}\tilde{S}_T &&= \mathbf{V}^{-1}\psi_{T,n}\mathbf{V} \qquad n\in\{2\ldots 2p\} \\
\tilde{\mathbf{S}}_T^{-1}(\psi_{T,1}) &= i(-1)^{p-1}\tilde{S}_T^{-1}\psi_{T,n}\tilde{S}_T &&= \mathbf{V}^{-1}\psi_{T,1}\mathbf{V}
\end{aligned}
$$

which is identical to the shift automorphism constructed in [14]. Following proposition 3.4.6 we know that the construction of the shift automorphism $\tilde{S}$ and the evolution automorphism given by the conjugation with $\tilde{R}$ is sufficient for constructing a hamiltonian quantum evolution in the sense of the previous sections.

Hence a nice side effect of the study of the above quantum lattice model of sine-Gordon type is the detection of a relation to another quantum lattice model, namely the massive Thirring model in it's reduced version as describing free massive fermions, as given by [14]. Since relations between these two models are known for the continous case, see e.g. [36, 11, 47] it seems to speak for the self coherence of the above lattice models, that they also exist in the discrete case.

## 3.5 Integrals of motion

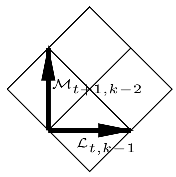
Using the quantum zerocurvature condition (3.3.6) define the quantum operators (k-t even)

$$\begin{array}{lllll} \mathcal{L}_{t,k-1} & := & L_{t,k}L_{t,k-1} & = & L_{t+1,k}L_{t+1,k-1} \\ \mathcal{M}_{t,k-1} & := & L_{t+1,k-1}L^{-1}_{t,k-1} & = & L^{-1}_{t+1,k}L_{t,k} \end{array} \qquad (3.5.78)$$

**Proposition 3.5.1** *[52] The matrices $\mathcal{L}_{i,j}$,$\mathcal{M}_{i,j}$ satisfy the following equation:*

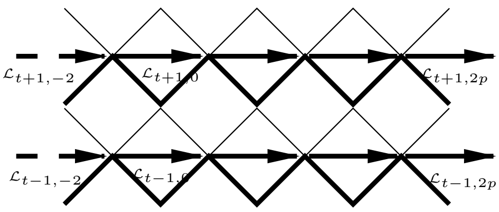
$$\mathcal{L}_{t+1,k} = \mathcal{M}_{t,k+1}\mathcal{L}_{t-1,k}\mathcal{M}^{-1}_{t,k-1} \qquad (3.5.79)$$

**Proof:**
The assertion follows immediately by using the quantum zero curvature equation (QZC) (3.3.6).
□
Equation (3.5.79) is called a **quantum Lax equation** for the evolution of quantum matrices assigned to the faces of the Minkowski space time lattice.




Now consider the matrix products of the two by two quantum matrices $\mathcal{L}_{i,j}$ (i-j always odd) along an elementary Cauchy zig zag of length $2p$ , where $2p$ is again the periodicity region of the face variables :

$$M_{2t+1,0} := \mathcal{L}_{2t+1,2p-2}\mathcal{L}_{2t+1,2p-4}\ldots\mathcal{L}_{2t+1,0} = \mathcal{M}_{2t,2p-1}M_{2t-1,0}\mathcal{M}^{-1}_{2t,-1} \qquad (3.5.80)$$



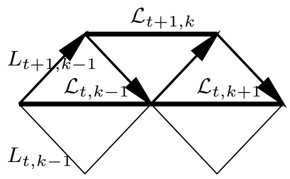

The two by two matrix $M_{i,j}$ is called a quantum monodromy matrix.
Let the edge operators be periodic, i.e. in particular $\mathcal{M}_{2t+1,2p-1} = \mathcal{M}_{2t+1,-1}$ and the commutation relations between them be ultralocal (case $b = 0, c = 2a$ in 5.1.11). Define

$$M_{t,0}(\lambda) := \begin{pmatrix} A_{t,0}(\lambda) & B_{t,0}(\lambda) \\ C_{t,0}(\lambda) & D_{t,0}(\lambda) \end{pmatrix} \qquad (3.5.81)$$

Using (3.5.80) we obtain a quantum integral of motion by taking traces of the monodromy matrices

$$m_{2t+1,0} := tr\, M_{2t+1,0} = tr\, M_{2t-1,0} = A_{2t-1,0} + D_{2t-1,0}$$

Analogous for even times.
Using again the QZC one obtains furthermore




$$\mathcal{L}_{t+1,k} = L_{t+1,k+1}\mathcal{L}_{t,k-1}L^{-1}_{t+1,k-1} = L^{-1}_{t+1,k+2}\mathcal{L}_{t,k+1}L_{t+1,k}.$$

Hence

$$\begin{aligned} M_{2t+1,0} &:= \mathcal{L}_{2t+1,2p-2}\mathcal{L}_{2t+1,2p-4}\dots\mathcal{L}_{2t+1,0} \\ &= L_{2t+1,2p-1}\underbrace{\mathcal{L}_{2t,2p-3}\dots\mathcal{L}_{2t,-1}}_{M_{2t,-1}}L^{-1}_{2t+1,-1} \\ &= L^{-1}_{2t+1,2p}M_{2t,1}L_{2t+1,0}. \end{aligned}$$

It follows that the trace of the monodromy matrix is translational invariant, i.e

$$m_{2t} := m_{2t,-1} = m_{2t,1} \Leftrightarrow trM_{2t,1} = trM_{2t,-1}$$

and that

$$m_{2t+1} = m_{2t} \Leftrightarrow trM_{2t+1,0} = trM_{2t,1},$$

which means that the trace of the monodromy matrix is invariant under light cone shifts.

**Theorem 3.5.2** *[52] If the edge algebra is formed by periodic generators* $\{(U_{t,k})_{k\in\{0,\dots 2p\}}, (V_{t,k})_{k\in\{0,\dots 2p\}}\}$ *($t \in \mathbb{Z}$ fixed), which obey ultralocal commutation relations then the trace of the monodromy matrix is expressable in terms of the face variables* $\{(P_{2t-1,2k})_{k\in\{0\dots p-1\}}(P_{2t,2k+1})_{k\in\{0\dots p-1\}}\}$ *or* $\{(P_{2t+1,2k})_{k\in\{0\dots p-1\}}(P_{2t,2k+1})_{k\in\{0\dots p-1\}}\}$ *respectively.*

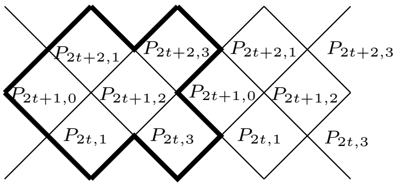
Since we explicitly defined light cone shifts within the algebra generated by the face operators (compare 3.4.6) we have henceforth found an automorphism on this algebra which leaves the traces of the monodromies invariant.

In terms of a physical interpretation the above light cone shifts can be understood as a fourier transform followed by time evolution.

In the following we will study the above introduced integrals in the simplest case, namely for $p = 1$.

## 3.6 The quantum pendulum

In the case $p = 1$ we have the following situation :




Choose $t =$ odd. Let $L_{t,k}$ and $P_{t,k-1}$ be defined as before (3.3.6,3.3.5).

$$\begin{aligned} trL_{t,3}L_{t,2}L_{t,1}L_{t,0} &= (e^{-i\frac{\alpha_t}{2}-i\frac{\alpha_{t-1}}{2}} + h.c.) + e^{4\lambda}(e^{i\frac{\alpha_t}{2}-i\frac{\alpha_{t-1}}{2}} + h.c.) \\ &- e^{2\lambda}(e^{-i\frac{\alpha_t}{2}-i\frac{\alpha_{t-1}}{2}}(P_{t,2} + P_{t,0} + (1+e^{i\alpha_t})P_{t-1,1} + kq^{\frac{m}{2}}P_{t-1,1}P_{t,2} \\ &+ \frac{1}{k}P_{t,0}P_{t-1,1}) + h.c.) \end{aligned} \tag{3.6.82}$$

where $P_{t,0}P_{t,2} = e^{i\alpha_t}$, $P_{t-1,1}P_{t-1,3} = e^{i\alpha_{t-1}}$ and $h.c.$ stands for the hermitian conjugate. For homogenity in time we demand that

$$\alpha_t = \alpha_{t-1} \Leftrightarrow P_{t,0}P_{t,2} = P_{t-1,1}P_{t-1,3} \tag{3.6.83}$$

hence $\alpha_T = \alpha = const.$ for all $T \in \mathbb{Z}$, since $\alpha_T = \alpha_{T+2}$, $\alpha_{T-1} = \alpha_{T+1}$ for $T \in \mathbb{Z}$, by the evolution of the face operators.
We consider again the root of unity case. Let $P^B_{t,0} = P^B_{t,2}$ and $P^B_{t,0} = P^B_{t,2}$, so that the evolution automorphism is given via conjugation with an evolution matrix (compare (3.3.43)). Define ($t =$ odd):

$$\begin{aligned} P_{t-1,1} &= e^{i\alpha}Q_{t-1} & P_{t-1,3} &= e^{i\alpha}Q^{-1}_{t-1} \\ P_{t,0} &= e^{i\alpha}Q_t & P_{t,2} &= e^{i\alpha}Q^{-1}_t \end{aligned}$$

Following the construction in the previous section the evolution automorphism is given by the conjugation with the matrices:

$$\mathbf{R}_t := R_k(q^{\frac{m}{2}}e^{i\frac{\alpha}{2}}Q_t)R_k(q^{\frac{m}{2}}e^{i\frac{\alpha}{2}}Q^{-1}_t) \quad \mathbf{R}_{t-1} := R_k(q^{\frac{m}{2}}e^{i\frac{\alpha}{2}}Q^{-1}_{t-1})R_k(q^{\frac{m}{2}}e^{i\frac{\alpha}{2}}Q_{t-1})$$

for even and odd time steps respectively. The shift automorphism is given by conjugation with the matrix

$$S_t := R_0(e^{i\frac{\alpha}{2}}Q_{t-1})R_0(e^{i\frac{\alpha}{2}}Q^{-1}_t)R_0(e^{i\frac{\alpha}{2}}Q^{-1}_{t-1})$$

and hence

$$S_tQ_{t-1}S^{-1}_t = Q_t.$$

Define $\mathbf{U} := \mathbf{R}_tS_t$. Following proposition 3.4.6 we find

$$Q_{t+n} = \mathbf{U}^nQ_t\mathbf{U}^{-n}.$$

The evolution equation reads as ($t$ now arbitrary):

$$Q_{t+1} = \frac{k + q^{\frac{m}{2}} e^{i\frac{\alpha}{2}} Q_t}{1 + kq^{\frac{m}{2}} e^{i\frac{\alpha}{2}} Q_t} \frac{k + q^{\frac{m}{2}} e^{-i\frac{\alpha}{2}} Q_t}{1 + kq^{\frac{m}{2}} e^{-i\frac{\alpha}{2}} Q_t} \tag{3.6.84}$$

The only nontrivial term in (3.6.83) is the $e^{2\lambda}$ term, which will be called $H_k(\alpha)$:

$$\begin{aligned} H_k(\alpha) &= 2\cos\frac{\alpha}{2}(Q_{t-1} + Q_{t-1}^{-1} + Q_t + Q_t^{-1}) + k(q^{\frac{m}{2}} Q_{t-1} Q_t + q^{-\frac{m}{2}} Q_t^{-1} Q_{t-1}^{-1}) \\ &+ \frac{1}{k}(q^{\frac{m}{2}} Q_t^{-1} Q_{t-1} + q^{-\frac{m}{2}} Q_{t-1}^{-1} Q_t). \end{aligned} \tag{3.6.85}$$

The equation (3.6.84) for the case $\alpha = 0$ can be viewed as a discrete version of the pendulum equation

$$Q_{tt} = 4\sin Q.$$

therefore the above sytem is called a quantum pendulum. The integral (3.6.85) is called hamiltonian of the quantum pendulum. It is related to an important hamiltonian appearing in solid state physics, called the Hofstadter hamiltonian (see e.g. [31, 33, 55, 17, 18, 3, 37, 35])

**Proposition 3.6.1** *Let $H_{Hof} = T + T^* + \frac{1}{k}(S + S^*)$ be the Hamiltonian of the Hofstadter model with $ST = e^{i\gamma} TS; \alpha = i\gamma$, then with the Substitutions:*

$$Q_n = e^{i\gamma/2} TS \quad \textit{and} \quad Q_{n-1} = e^{-i\gamma/2} TS^*$$

*one has:*

$$kH_{Hof}^2 - (k + \frac{1}{k}) = H_k(\gamma) \tag{3.6.86}$$

The relation to the "real pendulum" can also be found by reconstructing phase space via Husimi functions. The below pictures are done in the following way. First we construct a coherent state $\Psi_{00}$, which is characterized by the fact that the probability distributions of both observables $Q_n$ and $Q_{n-1}$ are maximally localized around $Q = 1$. The state $\Psi_{00}$ can be obtained as a solution of the following variational problem. First one should consider a set $S_a$ of states with a fixed value of Re$< \Phi|Q_n|\Phi >$

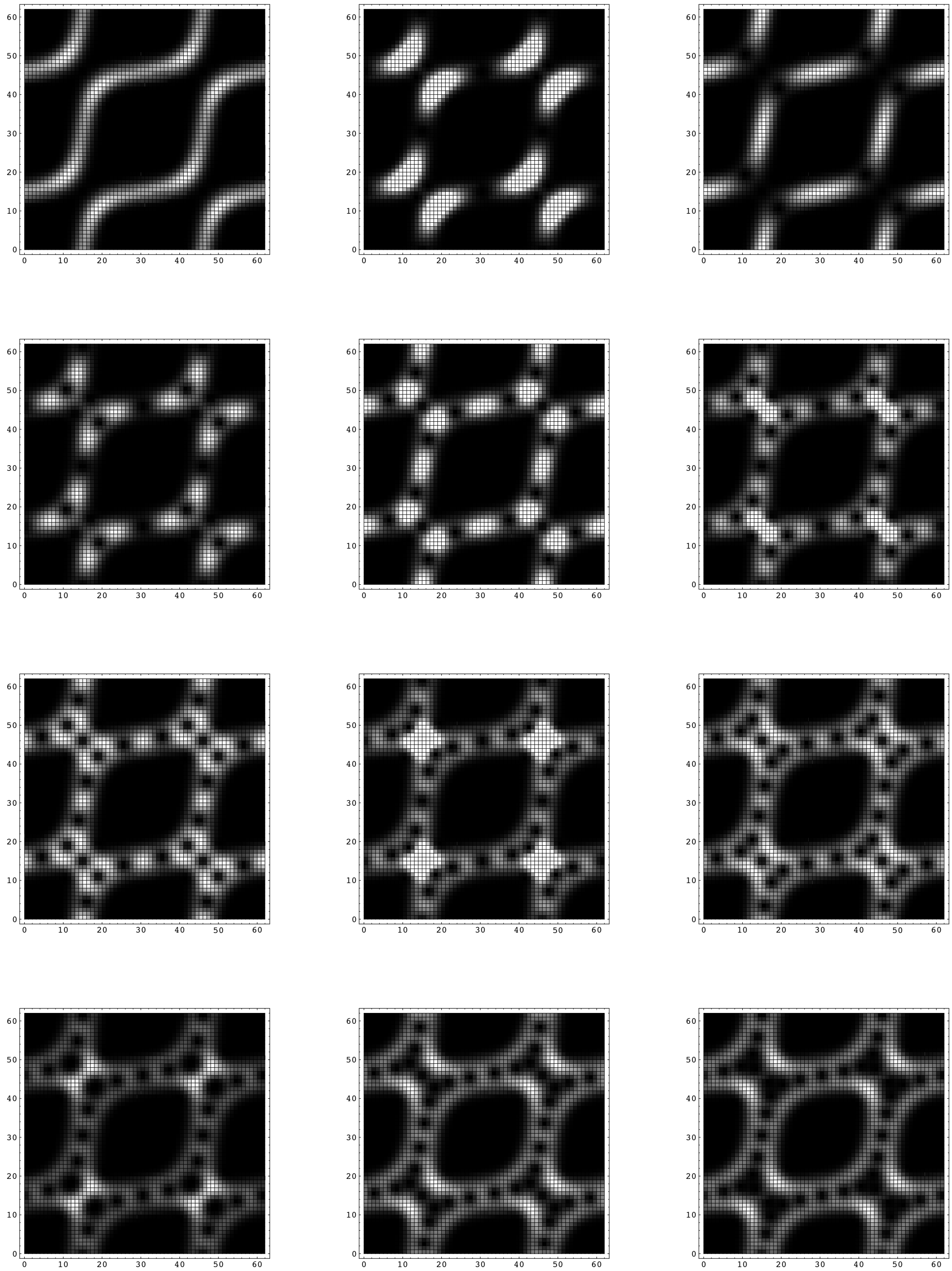
$$S_a = \{\Phi \in \mathcal{H} | < \Phi, \Phi >= 1, \mathrm{Re} < \Phi|Q_n|\Phi >= a\}.$$

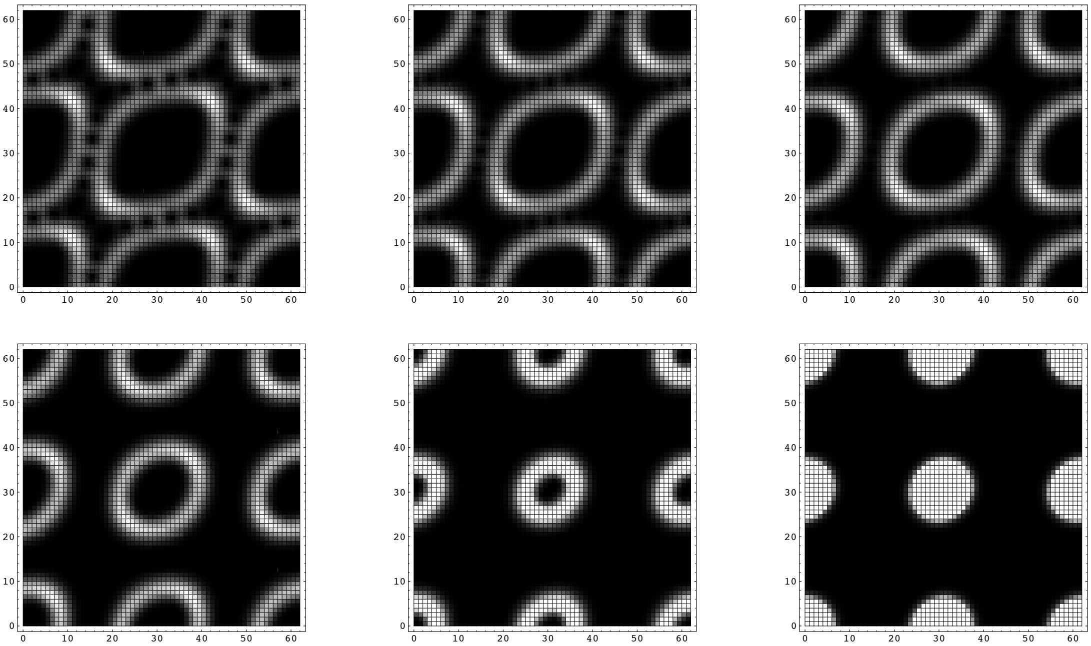
Here $\mathcal{H}$ is the Hilbert space, which is $\mathcal{H} = \mathbb{C}^N$ in our case. The maximum

$$b = \max_{\Phi \in S_a} \mathrm{Re} < \Phi|Q_{n-1}|\Phi >$$

is achieved at some state $\Phi^a$, which is equal to $\Psi_{00} := \Phi^a$, when $b = a$. In fact, $\Psi_{00}$ is the ground state of $-(Q_n + Q_n^* + Q_{n-1} + Q_{n-1}^*)$.

Then we obtain states $\Psi_{kl} = Q_n^k Q_{n-1}^l \Psi_{00}$ which are localized around the point $(e^{2\pi ik/N}, e^{2\pi il/N})$ in the classical phase space $S^1 \times S^1$. To make a picture of an eigenstate $\Psi \in \mathbb{C}^N$ we then display the "Husimi function" $(k, l) \mapsto |\langle \Psi, \Psi_{kl} \rangle|^2$ by grey levels. The pictures are done for the case $k = 0.7$ and $N = 31$ in ascending order.





## 3.7 On the spectrum of the quantum pendulum

### 3.7.1 The method of Bethe ansatz

The determination of the spectrum of the quantum pendulum hamiltonian, as well as the Hofstadter hamiltonian is not straight forward. In fact only a very few analytic solutions to this eigenvalue problem are known.

The following construction would give the Eigenstates and Eigenvalues of the hamiltonian of the quantum pendulum, if one would and could solve a set of algebraic equations - the socalled Bethe ansatz equations. The technique of Bethe ansatz was developped in numerous papers, starting with a paper of Hans Bethe [4]. References to the ansatz used here can be found e.g. in [17, 32, 55, 50].

As in the last section we will consider the case of periodic edge operators obeying ultralocal commutation relations (case $b = 0$, $c = 2a$ in (5.1.11).

The operators $L_{t,k}(\lambda)$ (see (3.3.6)) satisfy the socalled **fundamental commutation relations**:

$$R(\lambda-\mu)L_{t,k}(\lambda)\otimes L_{t,k}(\mu) = L_{t,k}(\mu)\otimes L_{t,k}(\lambda)R(\lambda-\mu) \tag{3.7.87}$$

where $R(\lambda)$ is the R-matrix of the Sine-Gordon model (see e.g. [32, 46, 51]):

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{rcl} b(\lambda-\mu) & = & \frac{e^{-\alpha}-e^{\alpha}}{e^{\mu-\lambda-\alpha}-e^{\lambda-\mu+\alpha}}; \\ c(\lambda-\mu) & = & \frac{e^{\mu-\lambda}-e^{\lambda-\mu}}{e^{\mu-\lambda-\alpha}-e^{\lambda-\mu+\alpha}} \end{array}$$

As a consequence the traces of the monodromy matrix $M_t(\lambda)$ commute for different values of $\lambda$:

$$\begin{aligned} & tr(R(\lambda-\mu)L_{t,2p-1}\dots L_{t,0}(\lambda)\otimes L_{t,2p-1}(\mu)\dots L_{t,0}(\mu)R(\lambda-\mu)^{-1}) \\ = \; & trM_t(\lambda)\cdot trM_t(\mu) \overset{(3.7.87)}{=} trM_t(\mu)trM_t(\lambda). \end{aligned}$$

Since the matrices $L_{t,k}$ are ultralocal and periodic we can choose a representation such that

$$L_{t,k} =: \mathbf{1} \otimes \dots \mathbb{1}\, L_t \otimes \mathbf{1} \dots \mathbb{1}$$

where $L$ acts on a local Hilbert space $\mathfrak{h}_k = \mathfrak{h}$ and $L_{t,k}$ on "big" Hilbert space $\bigotimes_{k=0}^{2p-1} \mathfrak{h}_k$.
Assume there exists a state $\omega_t \in \mathfrak{h}$ such that

$$L_t\omega_t =: \begin{pmatrix} \mathbf{a_t} & \mathbf{b_t} \\ \mathbf{c_t} & \mathbf{d_t} \end{pmatrix} \omega_t = \begin{pmatrix} a_t & \mathbf{b_t} \\ 0 & d_t \end{pmatrix} \omega_t$$

where $a$ and $d$ shall be the eigenvalues of the operators $\mathbf{a}$ and $\mathbf{d}$, respectively.
Let $\Omega = \bigotimes_{k=0}^{2p-1} \omega$, then (3.5.81)

$$M_t(\lambda) = L_{t,2p-1}(\lambda)\dots L_{t,0}(\lambda) = \begin{pmatrix} A_t(\lambda) & B_t(\lambda) \\ C_t(\lambda) & D_t(\lambda) \end{pmatrix} = \begin{pmatrix} a_t^{2p}(\lambda) & B_t(\lambda) \\ 0 & d_t^{2p}(\lambda) \end{pmatrix}.$$

Hence if there exists a state $\omega$ then the construction of an eigenvalue of $A$ and $D$ becomes considerably easy.

Moreover the fundamental commutation relations (short FCR) given in (3.7.87) gives us commutation relations between the operators $\mathbf{a_t}, \mathbf{b_t}, \mathbf{c_t}, \mathbf{d_t}$, which include in particular the following two equations:

$$\begin{aligned} \mathbf{a_t}(\lambda)\mathbf{b_t}(\mu) &= \frac{1}{c(\mu-\lambda)}(\mathbf{b_t}(\mu)\mathbf{a_t}(\lambda) - b(\mu-\lambda)\mathbf{b_t}(\lambda)\mathbf{a_t}(\mu)) \\ \mathbf{d_t}(\lambda)\mathbf{b_t}(\mu) &= \frac{1}{c(\lambda-\mu)}(\mathbf{b_t}(\mu)\mathbf{d_t}(\lambda) - b(\lambda-\mu)\mathbf{b_t}(\lambda)\mathbf{d_t}(\mu)) \qquad (3.7.88) \end{aligned}$$

These relations hold of course also for the elements $A, B, C, D$ of the monodromy operator. Hence using

$$\frac{b(\mu-\lambda)}{c(\mu-\lambda)} = -\frac{b(\lambda-\mu)}{c(\lambda-\mu)}$$

and (3.7.88)one sees that

$$(A_t(\lambda) + D_t(\lambda))B_t(\mu)\Omega_t = B_t(\mu)(A_t(\lambda)+D_t(\lambda))\Omega_t = (a_t^{2p}(\lambda) + d_t^{2p}(\lambda))B_t(\mu)\Omega_t,$$

if there exists a $\mu \in \mathbb{C}$ such that

$$\frac{b(\mu-\lambda)}{c(\mu-\lambda)}((A_t(\mu) - D_t(\mu))\Omega_t = 0.$$

I. e. in this case $B_t(\mu)\Omega$ is again an eigenstate of the trace of the monodromy matrix. Repeated use of the above gives finally that

$$(A_t(\lambda)+D_t(\lambda))B_t(\lambda_1)B_t(\lambda_2)\dots B_t(\lambda_n)\Omega_t = \Lambda(\lambda,\lambda_1,\lambda_2\dots\lambda_n)\Omega_t$$

where

$$\Lambda(\lambda,\lambda_1,\lambda_2\dots\lambda_n) = a_t^{2p}(\lambda)\prod_{l=1}^{2p}\frac{1}{c(\lambda_l-\lambda)} + d_t^{2p}(\lambda)\prod_{l=1}^{2p}\frac{1}{c(\lambda-\lambda_l)}$$

and $\lambda_1,\lambda_2\dots\lambda_n$ are solutions to the **Bethe ansatz equations**:

$$\prod_{l=2}^{n}\frac{c(\lambda_j-\lambda_l)}{c(\lambda_l-\lambda_j)} - \frac{d_t^{2p}}{a_t^{2p}} = 0. \qquad (3.7.89)$$

For more details see please [50]. In the following section we will construct a vacuum state $\Omega_t$ and Bethe ansatz equations for the quantum pendulum with special values of the parameter $\alpha$.

### 3.7.2 Bethe ansatz for the quantum pendulum

Let

$$e^{i\Pi} = e^{i\eta}\begin{pmatrix} 0 & 0 & . & . & 0 & 1 \\ 1 & 0 & & . & 0 & 0 \\ 0 & 1 & 0 & & & 0 \\ . & & . & . & & . \\ . & & & . & . & . \\ 0 & & & & 1 & 0 \end{pmatrix} \qquad e^{i\Phi} = e^{i\theta}\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & q & 0 & & & . \\ . & 0 & q^2 & 0 & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & q^{n-1} \end{pmatrix}$$

be $B$ times $B$ matrices acting on Hilbert space $\mathbb{C}^B$ provided with an orthogonal basis by the vectors $e_i = (0,0,\dots,1,\dots,0)^T$. Let

$$e^{i\Pi_1} = e^{i\Pi}\otimes \mathbb{1} \qquad e^{i\Pi_2} = \mathbb{1}\otimes e^{i\Pi}$$

analogous for $e^{i\Phi}$. By the ultralocal commutation relations of the edge operators $e^{i\Phi_l}$ and $e^{i\Pi_l}$ , $i=1,2$ are possible representations for the following operators along a Cauchy zig zag $\mathcal{C}_t$ :

$$\begin{aligned} e^{i\Pi^{Op}_{t,k}} &= U_{t,k}V^{-1}_{t,k-1} \\ e^{i\Phi^{Op}_{t,k}} &= U^{-1}_{t,k}U^{-1}_{t,k-1}, \end{aligned}$$

hence we will from now on skip the index "$Op$" . For simplicity we will also skip the time index, if clear what is meant . Note that $q^{-4a} = q^{-\frac{m}{2}} = q$. The operator $O_{t,k-1}$ defined in (3.3.5):

$$e^{ia_{t,k}} = O_{t,k-1} = V^{-1}_{t,k}V_{t,k-1}U_{t,k}U_{t,k-1}$$

commutes with all elements $e^{i\Pi_{t,k}}$ and $e^{i\Phi_{t,k}}$ and will be henceforth treated as a number. Taking the product of two L-matrices we find ($n-t$ even)

$$L_{t,n}(\lambda)L_{t,n-1}(\lambda) = \begin{pmatrix} e^{-i\Phi_n} - e^{2\lambda}e^{i\Phi_n+ia} & -e^{i\Pi_n}h_+(e^{i\Phi_n},a) \\ h_-(e^{i\Phi_n},a)e^{-i\Pi_n} & e^{i\Phi_n} - e^{2\lambda}e^{-i\Phi_n-ia} \end{pmatrix} \tag{3.7.90}$$

with

$$\begin{array}{rcl} h_+(e^{i\Phi_n},a) & = & e^{\lambda+\omega} + e^{\lambda-\omega}e^{2i\Phi_n+a}q^{-1} \\ h_-(e^{i\Phi_n},a) & = & e^{\lambda+\omega} + e^{\lambda-\omega}e^{-2i\Phi_n-a}q, \quad \text{where } a \in \mathbb{R}; \lambda \in \mathbb{C} \end{array}$$

The relation to the monodromy of face variables is given by (cf. (3.6.83)

$$\begin{array}{rcl} P_{t-1,1}P_{t-1,3} & = & e^{-i(a_1+a_2)} \\ P_{t,0}P_{t,2} & = & e^{2i\Phi_1+2i\Phi_2+ia_1+ia_2} \end{array} \tag{3.7.91}$$

That means that although $P_{t,0}P_{t,2}$ is a Casimir within the algebra generated by the face operators, it is not a Casimir within the algebra generated by the variables $e^{i\Pi_n}$; $e^{i\Phi_n}$, $i=1,2$.

The monodromy matrix of the quantum pendulum was given by the product of four L-matrices.

$$M(\lambda) = L_4(\lambda)L_3(\lambda)L_2(\lambda)L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

Hence we need to find a state $\Omega \in \mathbb{C}^B$ which gets annihilated by the operator

$$\begin{array}{rcl} C(\lambda) & = & -e^{2\lambda}(h_-(e^{i\Phi_2},a_2)e^{-i\Pi_2+i\Phi_1+ia_1} + e^{-i\Phi_2-ia_2}h_-(e^{i\Phi_1},a_1)e^{-i\Pi_1}) \\ & & -e^{0}(h_-(e^{i\Phi_2},a_2)e^{-i\Pi_2-i\Phi_1} + e^{i\Phi_2}h_-(e^{i\Phi_1},a_1)e^{-i\Pi_1}). \end{array}$$

Let us make the following ansatz for a vacuum state

$$\Omega := \sum_{j=0}^{B-1} f(q^j)e_j \otimes e_{-j-z+1},$$

hence we get the condition

$$e^{-ia_1-ia_2} = e^{2i\theta_1+2i\theta_2-2z} \tag{3.7.92}$$

and the functional equation

$$\frac{f(q^j)}{f(q^{j+1})} = -q^{-z}e^{i\theta_1+i\theta_2}e^{i\eta_2-i\eta_1}\frac{h_-(e^{i\theta_1}q^j,a_1)}{h_-(e^{i\theta_2}q^{-j-z},a_2)}$$

i.e. with $x_j := e^{2i\theta_1+ia_1}q^{2j}$

$$\frac{f(x_j)}{f(x_{j+1})} = -q^{-z}e^{i\theta_1+i\theta_2}e^{i\eta_2-i\eta_1}x_j^{-1}\frac{\tilde{k}+x_j}{1+\tilde{k}x_j} \tag{3.7.93}$$

where $\tilde{k} = kq^{-1}$. Note that

$$P_{t,0}P_{t,2}\Omega = e^{2i\Phi_2+2i\Phi_1+ia_2+ia_1}\Omega = q^{-2z}e^{2i\theta_1+2i\theta_2+ia_2+ia_1+2}\Omega = q^2\Omega$$

by condition (3.7.92). On the other hand we have already demanded (3.6.83) that

$$P_{t,0}P_{t,2} = P_{t-1,1}P_{t-1,3} = e^{-(ia_1+ia_2)}$$

Hence the functional equation reduces with $e^{\frac{i\alpha_1+i\alpha_2}{2}} \overset{!}{=} q$ and condition (3.7.92) to

$$\frac{f(x_j)}{f(x_{j+1})} = -qe^{i\eta_2-i\eta_1}x_j^{-1}\frac{\tilde{k}+x_j}{1+\tilde{k}x_j}.$$

Which can be straightforward solved by use of theorem (3.3.5) if we restrict the factors $e^{i\theta}$ and $e^{i\eta}$ appropriately [38].Hence we have constructed a state $\Omega$ such that

$$C(\lambda,\rho)\Omega = 0.$$

Moreover

$$\begin{aligned} A(\lambda)\Omega &= e^{-i\theta_1-i\theta_2}q^{z-1} + e^{4\lambda}e^{i\theta_1+i\theta_2+ia_1+ia_2}q^{-z+1} + e^{2\lambda}(q^ze^{-i\theta_1-i\theta_2}(\frac{1}{k}+k)) \\ &= e^{\frac{ia_1+ia_2}{2}}(q^{-1}+e^{4\lambda}q+e^{2\lambda}(\frac{1}{k}+k)). \end{aligned}$$

Following the method of the algebraic Bethe ansatz, together with the fact that

$$A(\lambda) = \overline{D(\bar{\lambda})}.$$

we could solve the eigenvalue problem

$$(A(\lambda)+D(\lambda))B(\lambda_1)B(\lambda_2)...B(\lambda_k)\Omega = \Lambda(\lambda,\lambda_1,\lambda_2,...,\lambda_k)\Omega$$

if we could find $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{C}$ as solutions to the Bethe ansatz equations ($e^{2\lambda_i} =: z_i$):

$$\prod_{\substack{i\neq j\\ i=1}}^{k} \frac{z_jq - z_iq^{-1}}{z_jq^{-1}-z_iq} = e^{-ia_1-ia_2}\quad\frac{q+z_j^2q^{-1}+z(k+\frac{1}{k})}{q^{-1}+z_j^2q+z(k+\frac{1}{k})}. \tag{3.7.94}$$

Moreover it can be shown that with

$$\phi_l(\lambda_1,\lambda_2,...,\lambda_l) := B(\lambda_1)B(\lambda_2)...B(\lambda_l)\Omega$$

one has

$$\sqrt{P_{t,0}P_{t,2}}\phi_l(\lambda_1,\lambda_2,...,\lambda_l,\rho) = q^{-(l+1)}\phi_l(\lambda_1,\lambda_2,...,\lambda_l,\rho).$$

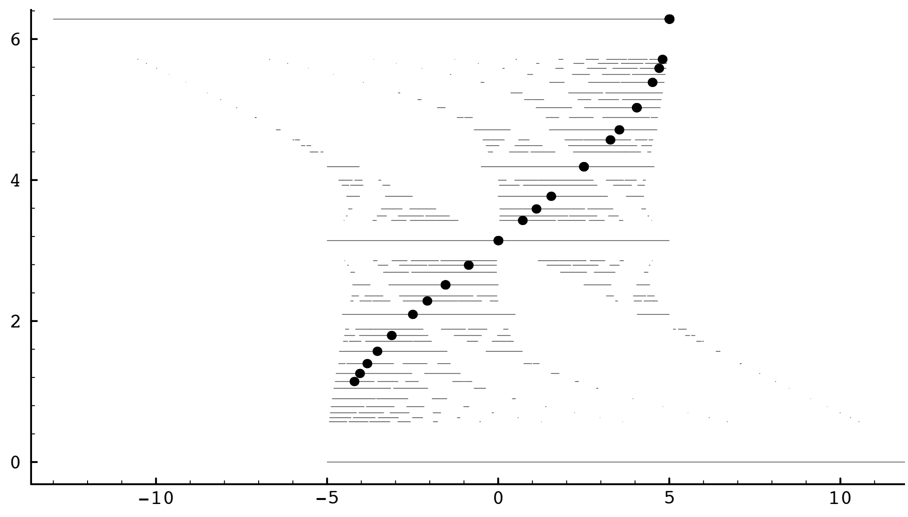
Consequently if we define $e^{\frac{i\alpha_l}{2}} := q^{-(l+1)}$ we know that $\phi_{l+Bj}(\lambda_1, \lambda_2, ..., \lambda_l)$ is a j'th eigenstate of

$$\begin{aligned} H &= 2\cos\frac{\alpha_l}{2}(Q_n + Q_n^* + Q_{n-1} + Q_{n-1}^*) + k(qQ_n^*Q_{n-1}^* + q^{-1}Q_{n-1}Q_n). \\ &+ \frac{1}{k}(q^{-1}Q_nQ_{n-1}^* + qQ_{n-1}Q_n^*) \end{aligned}$$

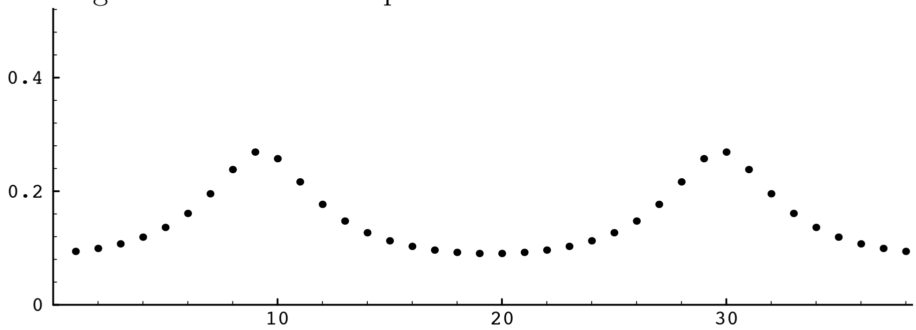
So the algebraic Bethe ansatz of the generalized Sine-Gordon model would - if one could solve the Bethe ansatz equations and if these are sufficient - give the spectrum of the Quantum pendulum for all $e^{i\frac{\alpha_l}{2}} = q^{-(l+1)}$.

Unfortunately e.g. the square lattice Hofstadter hamiltonian doesn't fall into this class, since there $e^{i\alpha} = q$.

In the following figure we see how the eigenvalues of $\Omega$ are distributed within the band spectrum of the Quantumpendulum for $k = 2$. On the y-axis are the values of $\gamma$, where $q = e^{i\gamma}$ and on the x-axis are the values of the energy given.



The eigenfunction $\Omega$ in a representation with $N = 41$ and $k = 2$.



# Chapter 4

# Other applications of the s-G equation

## 4.1 Elliptic billards and the pendulum equation

### 4.1.1 Planar billards

Let

$$\begin{array}{rccccl} x: & [0,2\pi) & \to & \mathbb{R}^2 & & \\ & t & \mapsto & x(t) & = & \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \end{array}$$

be a closed regular curve, algebraically given by the zero'th level set of a function $f : \mathbb{R}^2 \to \mathbb{R}$ , $f(u(t), v(t)) = 0$ with $grad\, f|_x \neq 0$ for all $x \in \mathbb{R}^2$. Assume that $grad\, f|_x \neq 0$ points to the outside of the curve for all times $t$. We imagine a force free point mass sliding inside the region in $\mathbb{R}$, which is bounded by $x$. Once in while the particle will elastically hit the boundary at a point $x_k = x(t_k)$ and will be reflected according to the law of preservation of momentum, which means in $\mathbb{R}^2$ that "incoming angle = outgoing angle" and that the absolut value of the momentum is preserved. The motion between two impacts is force free so the point mass will move on geodesics which means in $\mathbb{R}^2$ on straight lines.

Let us denote with $y_k$ the momentum of the point mass before the k'th impact and with $y_{k+1}$ its momentum after the k'th impact on the boundary. Since $|y_k| = |y_{k+1}|$ we set $|y_k| = const. = 1$, i.e. $y_k = \frac{x_k - x_{k-1}}{|x_k - x_{k-1}|}$ inside the billard.

**Definition 4.1.1** *The evolution of a particle hitting the boundary $N$ times is given by a sequence $\{(x_0, y_0), ..., (x_N, y_N)\}$, where $(x_0, y_0) \in \mathbb{R}^2 \times \mathbb{R}^2$ are fixed initial values and $(x_k, y_k)$ are determined via the the equations of motion:*

$$x_{k+1} - x_k = \mu_k\, y_{k+1} \tag{4.1.1}$$

$$y_{k+1} - y_k = \nu_k\, grad\, f|_{x_k}. \tag{4.1.2}$$

*The multipliers* $\mu_k$ *and* $\nu_k$ *have to be determined by the conditions (*$f$ *defined as above)*

$$\begin{aligned} |y_k| &= 1 & (4.1.3) \\ f(x_k) &= 0, \quad f.\ a. \quad x_k \in x & (4.1.4) \end{aligned}$$

*i.e.* $\mu_k$ *has to be a unique nontrivial (if existing) solution to*

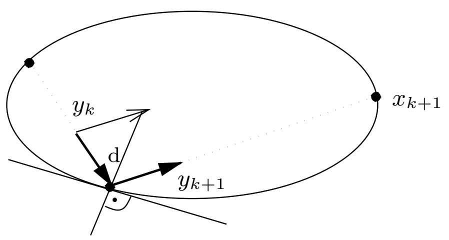
$$f(\mu_k y_{k+1} + x_k) = 0 \tag{4.1.5}$$

*and* $\nu_k$ *is defined by*

$$\nu_k = -2 \frac{\langle grad\, f|_{x_k}, y_k \rangle}{\langle grad\, f|_{x_k}, grad\, f|_{x_k} \rangle}, \tag{4.1.6}$$

*where* $\langle\rangle$ *is the usual euclidean scalar product.*

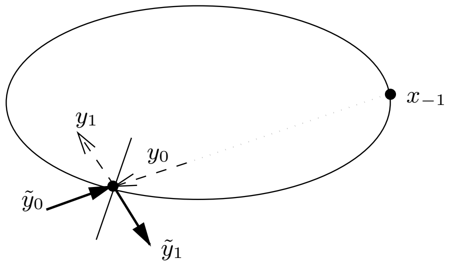
The first evolution equation (4.1.1) reflects the fact that the momentum is parallel to the difference of two consecutive impact locations (see Figure below). The second evolution equation (4.1.2) refers to the law of incoming angle equals outgoing angle, which means that the diagonal $d$ of the parallelogram spanned by the unit vectors $y_k$ and $y_{k+1}$ is perpendicular to the tangent to the boundary.




The above definition of a Billard evolution could be generalized to higher dimensions, which we will not need in the sequel. A higher dimensional extension of the above definition was proposed in [41] for the special case of the boundary being an ellipsoid. If the boundary is an ellipse then definition (4.1.1) and the one given in [41] are identical.

Besides the "unreal" assumptions in our model of a planar billard which are that the billard "particle" slides torsion- and frictionless over the billard plane and other physical idealizations, the billard of definition (4.1.1) inherits an additional "unphysical" feature. Let us explain this.

We observe that if one starts the billard evolution with a momentum $\tilde{y}_0$, which points from outside onto the billard boundary instead with a momentum $y_0 = -\tilde{y}_0$ emerging from a virtual location $x_{-1}$, i.e. $y_0 = \frac{x_0 - x_{-1}}{|x_0 - x_{-1}|}$.




then by (4.1.2) and (4.1.6)

$$\tilde{y}_1 + y_0 = -\nu_0\, gradf\, |_{x_0}, \text{ since } \tilde{\nu}_0 = -2\frac{\langle grad\, f|_{x_0}, \tilde{y}_0\rangle}{\langle grad\, f|_{x_0}, grad\, f|_{x_0}\rangle} = -\nu_0$$

hence

$$y_1 = -\tilde{y}_1.$$

From (4.1.5) we obtain as (per assumption) unique solutions to

$$\begin{aligned} f(\mu_0 y_1 + x_0) &= 0 \\ f(\tilde{\mu}_0 \tilde{y}_1 + x_0) &= 0 \end{aligned}$$

that

$$\tilde{\mu}_0 = -\mu_0$$

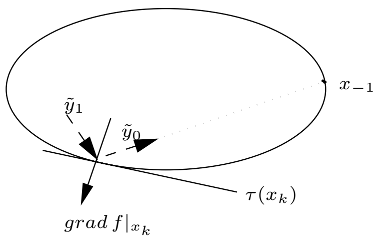
so finally with (4.1.1)

$$x_1 - x_0 = \mu_0 y_0 = \tilde{\mu}_0 \tilde{y}_0.$$

This means that the further evolution of the impact positions $x_k$ will be the same, regardless wether we start with the initial momentum $y_0$ or $\tilde{y}_0 = -y_0$.

Physically, starting with a momentum $\tilde{y}_0$, which points from outside onto the billard boundary doesn't seem to make too much sense. Hence in an appropriate physical model one would may be like to exclude such unphysical initial conditions. However this is not necessary, if one interprets the momentum $\tilde{y}_0$ not as resulting from the a particle hitting the boundary from outside, but rather as a backward in time notated momentum of a particle, which truly emerges from $x_{-1}$, i.e.

$$\tilde{y}_0 = \frac{x_{-1} - x_0}{|x_{-1} - x_0|}.$$




In short, since

$$\langle x_{k+1} - x_k, grad\, f|_{x_k}\rangle \leq 0 \quad \text{f.a.} \quad x_k \in x$$
$$\mu_k < 0 \Leftrightarrow \langle y_{k+1}, grad\, f|_{x_k}\rangle > 0.$$

### 4.1.2 Yet another definition of planar billards

We will from now on provide the boundary curve $x$ with an anticlockwise orientation, i.e. the inside of $x$ shall lying left to the oriented tangent $\tau$.
Let $x_k$ be a point on the boundary $x$. Let $y_{k+1}$ be an arbitrary unit vector on $\mathbb{R}^2$, which is nontangent to $x$ at $x_k$. If

$$(y_{k+1}, \tau(x_k)) \in (0, \pi)\ mod\ 2\pi$$

$$\Leftrightarrow$$

$$\langle y_{k+1}, grad\, f|_{x_k} \rangle < 0 \Leftrightarrow \mu_k > 0$$

then let $\{x^i_{k+1}\}_{i \in \{1,2\ldots M\}}$ be the distinct solutions to

$$y_{k+1} = \frac{x^i_{k+1} - x_k}{|x^i_{k+1} - x_k|}, \tag{4.1.7}$$

if

$$(y_{k+1}, \tau(x_k)) \in (\pi, 2\pi)\ mod\ 2\pi$$

$$\Leftrightarrow$$

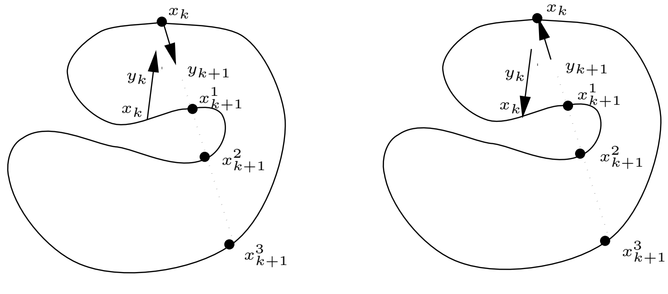
$$\langle y_{k+1}, grad\, f|_{x_k} \rangle > 0 \Leftrightarrow \mu_k < 0$$

then let $\{x^i_{k+1}\}_{i \in \{1,2\ldots M\}}$ be the distinct solutions to

$$y_{k+1} = \frac{x_k - x^i_{k+1}}{|x^i_{k+1} - x_k|}. \tag{4.1.8}$$

The above reflects the previously mentioned "unphysical" feature of our billard model.

We notice that if the region bounded by $x$ is convex then there usually exists one unique solution $x_{k+1}$, which refers to the next impact location. In general there will be more solutions $x^i_{k+1}$ to the above equations, i.e. more possible candidates for a next impact location.




forward in time  backward in time

Let us define:

$$l(x^i_{k+1}) := |x^i_{k+1} - x_k|;$$

Let

$$x_{k+1} := x^i_{k+1}$$

where $x^i_{k+1}$ is such that

$$l(x^i_{k+1}) = min\{l(x^j_{k+1})\}$$

for all solutions

$$x^j_{k+1} \in \{x^i_{k+1}\}_{i\in\{1,2...M\}}$$

to (4.1.8). Hence the next impact location $x_{k+1}$ is the solution $x^i_{k+1}$, which is closest to the starting location $x_k$. If $y_{k+1}$ is tangent to $x$ at the point $x_k$ then define $x_{k+1} := x_k$.
With the above described procedure we obtain for any pair $(x_k, y_{k+1})$, defined as above, a next impact location point $x_{k+1}$:

$$\begin{aligned} m : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 &(4.1.9)\\ (x_k, y_{k+1}) &\mapsto m((x_k, y_{k+1})) = x_{k+1} &(4.1.10)\end{aligned}$$

Soon the map $m$ will be used for defining an evolution on a planar billard, i.e. it will serve as a substitute for (4.1.1). But let us first study (4.1.2):

**Lemma 4.1.2** *The equation*

$$\langle y_{k+1} - y_k, \tau(x_k)\rangle = 0 \tag{4.1.11}$$

*$y_k, \tau, x$ as before is equivalent to (4.1.2) in the sense that for any given $(x_k, y_k)$ the solution $y_{k+1}$ is the same.*

**Proof:**

$$\langle y_{k+1} - y_k, \tau(x_k)\rangle = 0 \Rightarrow y_{k+1} - y_k = \nu_k grad\, f|_{x_k} \quad \text{for some} \quad \nu_k \in \mathbb{R}$$

Hence

$$|y_{k+1} - y_k|^2 = 2 - 2\langle y_{k+1}, y_k\rangle = \nu_k^2 |grad\, f|_{x_k}|^2$$

and

$$\langle y_{k+1}, y_k\rangle - 1 = \nu_k \langle grad\, f|_{x_k} y_k\rangle.$$

$$\Rightarrow \nu_k = 0 \vee \nu_k = -2\frac{\langle grad\, f|_{x_k}, y_k\rangle}{|grad\, f|_{x_k}|^2}$$

□

**Definition 4.1.3** *The evolution of a particle hitting the boundary $N$ times is given by a sequence* $\{(x_0, y_0),$
$..., (x_N, y_N)\}$*, where* $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ *are fixed initial values and* $(x_k, y_k)$ *are determined via the the equations of motion:*

$$\begin{aligned} x_{k+1} &= m(x_k, y_{k+1}) && (4.1.12) \\ y_{k+1} &= \nu_k grad\, f|x_k + y_k; \quad \nu_k = -2\frac{\langle grad\, f|_{x_k}, y_k\rangle}{\langle grad\, f|_{x_k}, grad\, f|_{x_k}\rangle} \\ \Leftrightarrow \\ & \langle y_{k+1} - y_k, \tau(x_k)\rangle = 0 && (4.1.13) \end{aligned}$$

$y_k$ *and* $x$ *as before,* $m$ *as defined at the begin of this section.*

**Proposition 4.1.4** *If* $x$ *is a convex billard boundary fullfilling the preliminaries of (4.1.1) then for any given initial condition* $(x_0, y_0)$ *the evolutions* $\{(x_0, y_0), ..., (x_N, y_N)\}$ *obtained through (4.1.1,4.1.2) and (4.1.12,4.1.13) are the same.*

**Proof:**
" $\Rightarrow$ "
Inserting $y_1 = \nu_0 grad\, f|x_0 + y_0$ into (4.1.5) we obtain a unique $\mu_0 \in \mathbb{R}$ s. th.

$$x_1 - x_0 = \mu_0 y_1$$

Hence if

$$\begin{array}{llll} \mu_0 > 0 & \Rightarrow & |x_1 - x_0| = \mu_0 |y_1| = \mu_1 & \Rightarrow \; y_1 = \frac{x_1 - x_0}{|x_1 - x_0|} \\ \mu_0 < 0 & \Leftrightarrow & |x_1 - x_0| = -\mu_0 |y_1| = -\mu_1 & \Leftrightarrow \; y_1 = \frac{x_0 - x_1}{|x_1 - x_0|} \end{array}$$

Since the solution $x_1$ is unique the assertion follows.
" $\Leftarrow$ "
trivial by use of the previous lemmatas.
□
Concluding we find that if $x$ is a nonconvex curve then it may happen that there exists no unique solution $\mu_{k+1}$ to the constraint (4.1.5): $f(\mu_{k+1} y_{k+1} + x_k) = 0$ of definition (4.1.1) [41]. So definition (4.1.3) is in this sense more general and should be used if one asks the question wether a given evolution could be interpreted as a billard evolution.

### 4.1.3 Making use of dim 2

In the following we will represent the billard evolution as defined in definition (4.1.3) in special coordinates, which are very much adapted to the case of a two-dimensional planar billard. In these coordinates the geometrical interpretation of the evolution equations will become more apparent.

Let

$$y_k = \begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} \qquad x_k = x(\phi_k) = \begin{pmatrix} u(\phi_k) \\ v(\phi_k) \end{pmatrix} \qquad \dot{x} = \frac{d}{d\phi}x(\phi) \tag{4.1.14}$$

then (4.1.13):

$$\langle y_{k+1} - y_k, \tau(x_k) \rangle = 0$$

$\Leftrightarrow$

$$(\cos\alpha_{k+1} - \cos\alpha_k)\dot{u} + (\sin\alpha_{k+1} - \sin\alpha_k)\dot{v} = 0$$

If $\alpha \in (-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi), n \in \mathbb{Z}$ then

$$x = \tan\alpha \quad \Leftrightarrow \quad e^{2i\alpha} = \frac{1+ix}{1-ix}; \quad x \in \mathbb{R}.$$

Hence if $\dot{u} \neq 0$ and $\frac{\alpha_{k+1}+\alpha_k}{2} \in (-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi)$ then one gets using (4.1.13) immediately the following equivalence:

$$\frac{\dot{v}}{\dot{u}} = \tan\frac{\alpha_{k+1}+\alpha_k}{2} \quad \Leftrightarrow \quad e^{i\alpha_{k+1}+i\alpha_k} = \frac{\dot{u}+i\dot{v}}{\dot{u}-i\dot{v}}(\phi_k)$$

Consequently in this case $\alpha := \frac{\alpha_{k+1}+\alpha_k}{2}$ is the angle of the tangent $\tau(x_k)$ with the positive $x$-Axis.
If $\dot{u} = 0$ then $\tau(x_k)$ is perpendicular to the $x$-Axis and depending on $\tau(x_k)$ $\frac{\alpha_{k+1}+\alpha_k}{2} = \pm\frac{\pi}{2}\, mod\, 2\pi$, so trivially if

$$\dot{u} = 0 \Rightarrow \frac{\dot{u}+i\dot{v}}{\dot{u}-i\dot{v}} = -1$$

The converse that if

$$\frac{\dot{u}+i\dot{v}}{\dot{u}-i\dot{v}} = -1 \Rightarrow \dot{u} = 0$$

is also true, s.th. we have the following lemma:

**Lemma 4.1.5** *For*

$$y_k = \begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} \qquad x_k = x(\phi_k) = \begin{pmatrix} u(\phi_k) \\ v(\phi_k) \end{pmatrix}$$

*the equations*

$$\langle y_{k+1} - y_k, \tau(x_k) \rangle = 0 \;\; and \quad e^{i\alpha_{k+1}+i\alpha_k} = \frac{\dot{u}+i\dot{v}}{\dot{u}-i\dot{v}}(\phi_k) \tag{4.1.15}$$

*are equivalent.*

In terms of the coordinates (4.1.14) the evolution equations in definition 4.1.3 read now as:

$$e^{i\alpha_{k+1}} = \frac{u(\phi_{k+1}) - u(\phi_k) + iv(\phi_{k+1}) - iv(\phi_k)}{|u(\phi_{k+1}) - u(\phi_k) + iv(\phi_{k+1}) - iv(\phi_k)|} = \frac{z(\phi_{k+1}) - z(\phi_k)}{|z(\phi_{k+1}) - z(\phi_k)|} \tag{4.1.16}$$

$$e^{i\alpha_{k+1}+i\alpha_k} = \frac{\dot{u} + i\dot{v}}{\dot{u} - i\dot{v}}(\phi_k) =: \frac{\dot{z}(\phi_k)}{\dot{\bar{z}}(\phi_k)} \tag{4.1.17}$$

An immediate consequence of equations (4.1.16) and (4.1.17) is that if $\phi_{k+1} \to \phi_k$, i.e. if the particle "slides" along the boundary then equation (4.1.16) and (4.1.17) become (modulo taking the right square root of (4.1.17)) equal to the equation of the unit tangent in terms of the parametrization of the curve:

$$e^{i\alpha(\phi_k)} = \frac{z(\phi_k)}{|z(\phi_k)|}.$$

### 4.1.4 Billards with an ellipse as boundary

**Lemma 4.1.6** *Any symmetric Matrix $M \in Gl(2, \mathbb{R})$*

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

*has two strictly positive eigenvalues iff $a > 0$ and $det\, M > 0$.*

**Proof:**
The eigenvalues of $M$ are

$$\lambda_{1/2} = \frac{1}{2}(a + d \pm \sqrt{(a+d)^2 - 4det\, M}$$

" $\Rightarrow$ " If

$$\lambda_{1/2} > 0 \Rightarrow a + d > +\sqrt{(a+d)^2 - 4det\, M} > 0 \Rightarrow 4det\, M > 0 \Rightarrow a > 0, d > 0$$

" $\Leftarrow$ "

$$\begin{aligned} a > 0, det\, M > 0 \Rightarrow 0 < det\, M < ad \quad &\Rightarrow \quad (a+d)^2 - 4det\, M > (a+d)^2 - 4ad \\ &\quad\;\;\; = (a-d)^2 \\ &\Rightarrow \quad \sqrt{(a+d)^2 - 4det\, M} \in \mathbb{R} \end{aligned}$$

Since

$$det\, M > 0 \Rightarrow \sqrt{(a+d)^2 - 4det\, M} < a+d \Rightarrow \lambda_1 > 0 \wedge \lambda_2 > 0, \text{ as } a > 0 \text{ and } d > 0.$$

□

**Lemma 4.1.7** *For any symmetric Matrix* $M \in Gl(2,\mathbb{R})$

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

*with both eigenvalues being strictly positive there exists a matrix* $g \in Gl(2,\mathbb{R})$*such that* $M = g^T g$

**Proof:**
Let

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad a > 0, \quad ad - b^2 > 0 \quad \Rightarrow d > 0$$

Define

$$\mathcal{A} = \sqrt{\frac{a}{2}} \qquad \mathcal{C} = \sqrt{\frac{a}{2}}$$
$$\mathcal{B} = \frac{1}{\sqrt{2a}}(b - det\, g) \qquad \mathcal{D} = \frac{1}{\sqrt{2a}}(b + det\, g)$$

then with

$$g = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

we get $g^T g = M$, whilst $det\, g = \sqrt{det\, M} > 0$, so $g \in Gl(2,\mathbb{R})$.

□

Let $M$ be a real symmetric matrix with positive eigenvalues. Hence the equation

$$\langle Mx, x\rangle - 1 = 0 \tag{4.1.18}$$

defines an ellipse centered around the origin.

**Proposition 4.1.8**

a.) *For any solution* $x \in \mathbb{R}^2$ *to (4.1.18), there exists a matrix* $g \in Gl(2,\mathbb{R})$ *such that* $x$ *can be parametrized by* $x = g^{-1}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}; \quad \phi \in [0, 2\pi).$

b.) *For any* $g \in Gl(2,\mathbb{R})$ *the curve* $x := g^{-1}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}; \quad \phi \in [0, 2\pi)$ *defines an ellipse centered around the origin.*

**Proof:**
Let $x$ be a solution to (4.1.18), then by Lemma (4.1.7) there exists a matrix $g \in Gl(2,\mathbb{R})$ with $M = g^T g$ now:

$$1 = \langle Mx, x\rangle = \langle g^T g x, x\rangle = \langle g^T g g^{-1} y, g^{-1} y\rangle = \langle y, y\rangle$$

$$\Rightarrow y = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \Rightarrow x = g^{-1} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}; \quad \phi \in [0, 2\pi)$$

On the other hand, if

$$x := g^{-1} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}; \quad \phi \in [0, 2\pi)$$

then define $M := g^T g \quad \Rightarrow \langle Mx, x \rangle = 1$ describes an ellipse centered around the origin, as $g^T g$ has only positive eigenvalues which can be seen by using (4.1.6).

□

Now after all this technical remarks let us finally find the evolution equations for a billard with an elliptic boundary. Let $g \in Gl(2, \mathbb{R})$ with

$$g = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix},$$

let

$$x(\phi) = \begin{pmatrix} u(\phi) \\ v(\phi) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

In complex notation we get:

$$z = u + iv = Ae^{i\phi} + Be^{-i\phi}$$

where

$$A = \frac{1}{2}(\mathcal{A} + \mathcal{D} - i(\mathcal{B} - \mathcal{C})) \qquad (4.1.19)$$

$$B = \frac{1}{2}(\mathcal{A} - \mathcal{D} + i(\mathcal{B} + \mathcal{C})) \qquad (4.1.20)$$

So from (4.1.12)

$$e^{i\alpha_{k+1}} = \frac{z(\phi_{k+1}) - z(\phi_k)}{|z(\phi_{k+1}) - z(\phi_k)|} = \frac{A(e^{i\phi_{k+1}} - e^{i\phi_k}) + B(e^{-i\phi_{k+1}} - e^{-i\phi_k})}{|A(e^{i\phi_{k+1}} - e^{i\phi_k}) + B(e^{-i\phi_{k+1}} - e^{-i\phi_k})|},$$

hence

$$e^{2i\alpha_{k+1}} = -\frac{Ae^{i\phi_{k+1}+i\phi_k} - B}{\bar{A} - \bar{B}e^{i\phi_{k+1}+i\phi_k}}. \qquad (4.1.21)$$

Inverting (4.1.21) and inserting into (4.1.13) we finally obtain the equations of motion for the billard with an elliptic boundary:

$$e^{i\phi_{k+1}+i\phi_k} = -\frac{\bar{A}e^{2i\alpha_{k+1}} - B}{A - \bar{B}e^{2i\alpha_{k+1}}} \qquad (4.1.22)$$

$$e^{i\alpha_{k+1}+i\alpha_k} = -\frac{Ae^{i\phi_k} - B}{\bar{A} - \bar{B}e^{i\phi_k}} \qquad (4.1.23)$$

**Theorem 4.1.9** *The motion on a billard with elliptic boundary is integrable and symplectic.*

**Proof:**
Taking the square of both equations we get:

$$e^{2i\phi_{k+1}+2i\phi_k} = (-\frac{\bar{A}e^{2i\alpha_{k+1}} - B}{A - \bar{B}e^{2i\alpha_{k+1}}})^2 = \frac{\bar{A}^2 e^{2i\alpha_{k+1}} + B^2 e^{-2i\alpha_{k+1}} - 2\bar{A}B}{A^2 e^{-2i\alpha_{k+1}} + \bar{B}^2 e^{2i\alpha_{k+1}} - A\bar{B}} \quad (4.1.24)$$

$$e^{2i\alpha_{k+1}+2i\alpha_k} = (-\frac{Ae^{i\phi_k} - B}{\bar{A} - \bar{B}e^{i\phi_k}})^2 = \frac{A^2 e^{2i\phi_k} + B^2 e^{-2i\phi_k} - AB}{\bar{A}^2 e^{-2i\phi_k} + \bar{B}^2 e^{2i\phi_k} - \bar{A}\bar{B}} \quad (4.1.25)$$

we see immediately that (4.1.24) and (4.1.25) are of the form as in Theorem (4.1.11).
□

### 4.1.5 Extending the billard model

We already showed that the discrete pendulum equation:

$$u_{2t+1} - u_{2t} + u_{2t-1} = f(u_{2t}) \quad (4.1.26)$$

with

$$f(u) = -i\ Ln(\frac{k_1 + \alpha x_t}{1 + k_1 \alpha x_t}\frac{k_2 + \alpha x_t}{1 + k_2 \alpha x_t}), \text{ where } x = e^{iu_t}$$

permits even in the quantum case an integral of motion. For the classical system $k_1, k_2$ need not necessarily to be real, there exists also an integral of motion if $k_1, k_2 \in \mathbb{C}$. It was found by Suris in 1989 [48].

We suggest an extension of the above classical model.

**Lemma 4.1.10** *The time evolution given by the following map on phase space:*

$$\begin{aligned} T : \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R} \\ (u_{2t+1}, u_{2t}) &= T(u_{2t-1}, u_{2t-2}) \end{aligned}$$

$$\begin{aligned} u_{2t+1} &= h_1(u_{2t}) - u_{2t-1} = h_1(h_2(u_{2t-1}) - u_{2t-2}) - u_{2t-1} \\ u_{2t} &= h_2(u_{2t-1}) - u_{2t-2}, \ \textit{where } h_i \in C^1(\mathbb{R}) \end{aligned} \quad (4.1.27)$$

*is area preserving, hence symplectic.*

**Proof:** The proof is straight forward computation:

$$\frac{\partial u_{2t+1}}{\partial u_{2t-1}}\frac{\partial u_{2t}}{\partial u_{2t-2}} - \frac{\partial u_{2t+1}}{\partial u_{2t-2}}\frac{\partial u_{2t}}{\partial u_{2t-1}} = -h_1'(u_{2t})h_2'(u_{2t-1}) + 1 + h_1'(u_{2t})h_2'(u_{2t-1}) = 1$$

□

**Remark:** The above map can also be viewed as the composition of two symplectic "half-time" mappings $T_1$ and $T_2$:

$$\begin{pmatrix} u_{2t+1} \\ u_{2t} \end{pmatrix} = T_1 \begin{pmatrix} u_{2t} \\ u_{2t-1} \end{pmatrix} \qquad \begin{pmatrix} u_{2t} \\ u_{2t-1} \end{pmatrix} = T_2 \begin{pmatrix} u_{2t-1} \\ u_{2t-2} \end{pmatrix}$$

$$\begin{array}{rcl} u_{2t} & = & u_{2t} \\ u_{2t+1} & = & h_1(u_{2t}) - u_{2t-1} \end{array} \qquad \begin{array}{rcl} u_{2t} & = & h_2(u_{2t-1}) - u_{2t-2} \\ u_{2t-1} & = & u_{2t-1} \end{array}$$

If $T_1 = T_2 = T$ then it clearly suffices to study the "half-time" evolution $T$ only.

In the following we show that there exist functions $h_i \in C^1((-\pi, \pi]))$ such that the above system is integrable. Let us restrict $u$ to $u_t \in (-\pi, \pi]$. The map $x_t = e^{iu_t}$ is now an isomorphism and we can for simplicity rewrite the evolution in a multiplicative way. Note that the phase space is now $S^1 \times S^1$. $u_t$ is now looked at as an angle.

**Theorem 4.1.11** *Let $x_t = e^{iu_t}, u_t \in \mathbb{R}$ i.e. $\bar{x}_t = \frac{1}{x_t}$. The time evolution shall be given by*

$$x_{2t+1} = \left(\frac{\bar{d}_2}{d_2}\frac{C\bar{a}_1 + Ba_1 + Bd_1x_{2t}^{\epsilon_1} + C\bar{d}_1\bar{x}_{2t}^{\epsilon_1}}{\bar{C}a_1 + \bar{B}\bar{a}_1 + \bar{B}\bar{d}_1\bar{x}_{2t}^{\epsilon_1} + \bar{C}d_1x_{2t}^{\epsilon_1}}\right)^{\frac{1}{\epsilon_2}} \quad \bar{x}_{2t-1} =: f_1(x_{2t})\bar{x}_{2t-1} \tag{4.1.28}$$

$$x_{2t} = \left(\frac{\bar{d}_1}{d_1}\frac{C\bar{a}_2 + \bar{B}a_2 + \bar{B}d_2x_{2t-1}^{\epsilon_2} + C\bar{d}_2\bar{x}_{2t-1}^{\epsilon_2}}{\bar{C}a_2 + B\bar{a}_2 + B\bar{d}_2\bar{x}_{2t-1}^{\epsilon_2} + \bar{C}d_2x_{2t-1}^{\epsilon_2}}\right)^{\frac{1}{\epsilon_1}} \quad \bar{x}_{2t-2} =: f_2(x_{2t-1})\bar{x}_{2t-2} \tag{4.1.29}$$

*with $d_i, a_i, B, C \in \mathbb{C}, \epsilon_1, \epsilon_2 \in \mathbb{R}$ are arbitrary, then*

$$\begin{array}{rcl} H(x_t, x_{t+1}) & = & \bar{C}w_1(x_{2t})w_2(x_{2t+1}) + C\bar{w}_1(x_{2t})\bar{w}(x_{2t+1}) \\ & + & Bw_1(x_{2t})\bar{w}_2(x_{2t+1}) + \bar{B}\bar{w}_1(x_{2t})w_2(x_{2t+1}) \end{array} \tag{4.1.30}$$

*where*

$$\begin{array}{rcl} w_1(x_{2t}) & = & a_1 + x_{2t}^{\epsilon_1}d_1 \\ w_2(x_{2t+1}) & = & a_2 + x_{2t+1}^{\epsilon_2}d_2 \end{array}$$

*is an integral of motion, i.e. $H(x_{2t}, x_{2t+1}) = H(x_{2t}, x_{2t-1}) = H(x_{2t-2}, x_{2t-1}) =$const.*

**Proof:** $w_2(x_{2t+1}) = w_2(x_{2t-1}) + d_2(x_{2t+1}^{\epsilon_2} - x_{2t-1}^{\epsilon_2})$, hence

$$H(x_{2t}, x_{2t+1}) = H(x_{2t}, x_{2t-1}) + \mathrm{R} \qquad \text{where}$$

$$\begin{array}{rcl} \mathrm{R} & = & \bar{C}w_1(x_{2t})(x_{2t+1}^{\epsilon_2} - x_{2t-1}^{\epsilon_2})d_2 + C\bar{w}_1(x_{2t})(\bar{x}_{2t+1}^{\epsilon_2} - \bar{x}_{2t-1}^{\epsilon_2})\bar{d}_2 \\ & + & \bar{B}\bar{w}_1(x_{2t})(x_{2t+1}^{\epsilon_2} - x_{2t-1}^{\epsilon_2})d_2 + Bw_1(x_{2t})(\bar{x}_{2t+1}^{\epsilon_2} - \bar{x}_{2t-1}^{\epsilon_2})\bar{d}_2 \end{array}$$

So $H(x_{2t}, x_{2t+1}) = H(x_{2t}, x_{2t-1})$ if $R = 0$. But $R = 0$ gives us the evolution (4.1.28):

$$-\frac{x_{2t+1}^{\epsilon_2} - x_{2t-1}^{\epsilon_2}}{\bar{x}_{2t+1}^{\epsilon_2} - \bar{x}_{2t-1}^{\epsilon_2}}\frac{d_2}{\bar{d}_2} = (x_{2t+1}x_{2t-1})^{\epsilon_2}\frac{d_2}{\bar{d}_2} = \frac{Bw_1(x_{2t}) + C\bar{w}_1(x_t)}{\bar{B}\bar{w}_1(x_{2t}) + \bar{C}w(x_{2t})}$$
$$= \frac{Ba_1 + C\bar{a}_1 + Bd_1x_{2t} + C\bar{d}_1\bar{x}_{2t}}{\bar{B}\bar{a}_1 + \bar{C}a_1 + \bar{B}\bar{d}_1\bar{x}_{2t} + \bar{C}d_1x_{2t}} \Leftrightarrow R = 0$$

In the same way using (4.1.29) we finally obtain $H(x_{2t}, x_{2t+1}) = H(x_{2t-2}, x_{2t-1})$
□

**Corollary 4.1.12** *The generalized discrete pendulum equation*

$$x_{t+1}x_{t-1} = \frac{k_1 + \alpha x_t}{1 + k_1\alpha x_t}\frac{k_2 + \bar{\alpha}x_t}{1 + k_2\bar{\alpha}x_t}, \quad k_1, k_2 \in \mathbb{R}, \alpha \in S^1$$

*is integrable.*

**Proof:** Let us for simplicity compute the integral for $k_1 = k_2$. From the above we get immediately that

$$B = 1, \quad C = k^2, \qquad \Rightarrow a = \frac{k(\alpha + \bar{\alpha})}{1 + k^2}$$

and hence a corresponding integral of motion is:

$$\frac{1}{k}H(x_{t-1}, x_t) - (1 + k^2)a^2 = (\alpha + \bar{\alpha})(x_t + \bar{x}_t + x_{t-1} + \bar{x}_{t-1}) \qquad (4.1.31)$$
$$+ \; k(x_tx_{t-1} + \bar{x}_t\bar{x}_{t-1}) + \frac{1}{k}(x_t\bar{x}_{t-1} + \bar{x}_tx_{t-1}).$$

□
If one redefines the evolution variables by multiplying them with a constant phase factor $x = ye^{i\phi}$ then $f_1^{new}(y) := f_1(ye^{i\phi_1})$ and $f_2^{new}(y) := f_2(ye^{i\phi_2})$ could in principle become identical mappings on $S^1$, i.e. $f_1^{new}(y) = f_2^{new}(y)$ for all $y \in S^1$. In this case the evolution will be of standard type, as described in [48]. The following lemma shows that there exist evolutions, which are nonstandard also if one takes the above redefinitions into consideration.

**Lemma 4.1.13** *There exist functions of the form $f_i : S^1 \to S^1$*

$$f_1(x) := \frac{A_1 + Dx + E\bar{x}}{\bar{A}_1 + \bar{D}\bar{x} + \bar{E}x} \qquad f_2(x) := \frac{A_2 + \bar{D}x + E\bar{x}}{\bar{A}_2 + D\bar{x} + \bar{E}x}$$

*such that the evolution given by*

$$x_{2t+1}x_{2t-1} = f_1(x_{2t})$$
$$x_{2t}x_{2t-2} = f_2(x_{2t-1})$$

*is different for even and odd variables, also if one redefines the evolution variables by*

$$\begin{aligned} y_{2t+1} &:= x_{2t+1}e^{-i\phi_1} \\ y_{2t} &:= x_{2t}e^{-i\phi_2} \end{aligned} \tag{4.1.32}$$

**Proof:**
Without losing generality we assume $D \neq 0, E \neq 0$. Since our evolution variables are $S^1$-valued we can rewrite our evolution functions $f_1$ and $f_2$ as:

$$f_1(x) := \frac{A_1 x + Dx^2 + E}{\bar{A}_1 x + \bar{D} + \bar{E}x^2} \qquad f_2(x) := \frac{A_2 x + \bar{D}x^2 + E}{\bar{A}_2 x + D + \bar{E}x^2}$$

hence with (4.1.32) we get:

$$\begin{aligned} y_{2t+1}y_{2t-1} &= \frac{e^{-2i\phi_1}(A_1 e^{i\phi_2}y_{2t} + De^{2i\phi_2}y_{2t}^2 + E)}{\bar{A}_1 e^{i\phi_2}y_{2t} + \bar{D} + \bar{E}e^{2i\phi_2}y_{2t}^2} =: \frac{g_1(y_{2t})}{\tilde{g}_1(y_{2t})} = f_1^{new}(y_{2t}) \\ y_{2t}y_{2t-2} &= \frac{e^{-2i\phi_2}(A_2 e^{i\phi_1}y_{2t-1} + \bar{D}e^{2i\phi_1}y_{2t-1}^2 + E)}{\bar{A}_2 e^{i\phi_1}y_{2t-1} + D + \bar{E}e^{2i\phi_1}y_{2t-1}^2} =: \frac{g_2(y_{2t-1})}{\tilde{g}_2(y_{2t-1})} = f_2^{new}(y_{2t-1}) \end{aligned}$$

If

$$\frac{g_1(y_{2t})}{\tilde{g}_1(y_{2t})} = \frac{g_2(y_{2t-1})}{\tilde{g}_2(y_{2t-1})} \quad \Rightarrow \quad g_1(y_{2t})\tilde{g}_2(y_{2t-1}) = g_2(y_{2t-1})\tilde{g}_1(y_{2t})$$

for any $y \in S^1$, we know by the Cauchy Integral Formula (since $g_1\tilde{g}_2$ and $g_2\tilde{g}_1$ are holomorphic) that the equality must also hold for any point inside the unit disc. In particular

$$\begin{aligned} &\Rightarrow g_1(0)\tilde{g}_2(0) = g_2(0)\tilde{g}_1(0) \\ &\Leftrightarrow e^{i\phi_1 - i\phi_2} = e^{i\phi_D} \end{aligned}$$

where $D = |D|e^{i\phi_D}$. Hence $e^{i\phi_1 - i\phi_2} = e^{i\phi_D}$ is a necessary condition for $f_1^{new}(y) = f_2^{new}(y), \quad y \in S^1$. The evolution satisfying this condition reads now as:

$$\begin{aligned} f_1^{new}(y_{2t}) &= \frac{A_1 e^{-i\phi_1} + |D|y_{2t} + Ee^{-i\phi_1 - i\phi_2}\bar{y}_{2t}}{\bar{A}_1 e^{i\phi_1} + |D|\bar{y}_{2t} + \bar{E}e^{i\phi_1 + i\phi_2}y_{2t}} \\ f_2^{new}(y_{2t-1}) &= \frac{A_2 e^{-i\phi_2} + |D|y_{2t-1} + Ee^{-i\phi_1 - i\phi_2}\bar{y}_{2t-1}}{\bar{A}_2 e^{i\phi_2} + |D|\bar{y}_{2t-1} + \bar{E}e^{i\phi_1 + i\phi_2}y_{2t-1}} \end{aligned}$$

But choosing $y_{2t} = y_{2t-1} = 1$ and then $y_{2t} = y_{2t-1} = -1$ as points on $S^1$ we see immediately that if $A_1\bar{A}_2 e^{i\phi_D} \neq \bar{A}_1 A_2 e^{i\phi_D}$ then $f_1^{new}(y) \neq f_2^{new}(y)$.
The same argument works if one makes the redefinitions:

$$\begin{aligned} y_{2t+1} &:= x_{2t+1}e^{-i\phi_1} \\ \bar{y}_{2t} &:= x_{2t}e^{-i\phi_2} \end{aligned} \qquad \begin{aligned} \bar{y}_{2t+1} &:= x_{2t+1}e^{-i\phi_1} \\ \bar{y}_{2t} &:= x_{2t}e^{-i\phi_2} \end{aligned}$$

□

## 4.2 $O^3$ Invariant Chiral Model, Neumann System

A weak Chebychev net in $S^2$ was a map (1.2.10) $N : \mathbb{Z} \longrightarrow S^2 \subset \mathbb{R}^3$ such that

$$\begin{array}{rclcl} \langle N_{n,m}, N_{n+1,m} \rangle & = & \langle N_{n,m+1}, N_{n+1,m+1} \rangle & = & \cos \delta_1 \\ \langle N_{n,m}, N_{n,m+1} \rangle & = & \langle N_{n+1,m}, N_{n+1,m+1} \rangle & = & \cos \delta_2 \end{array}$$

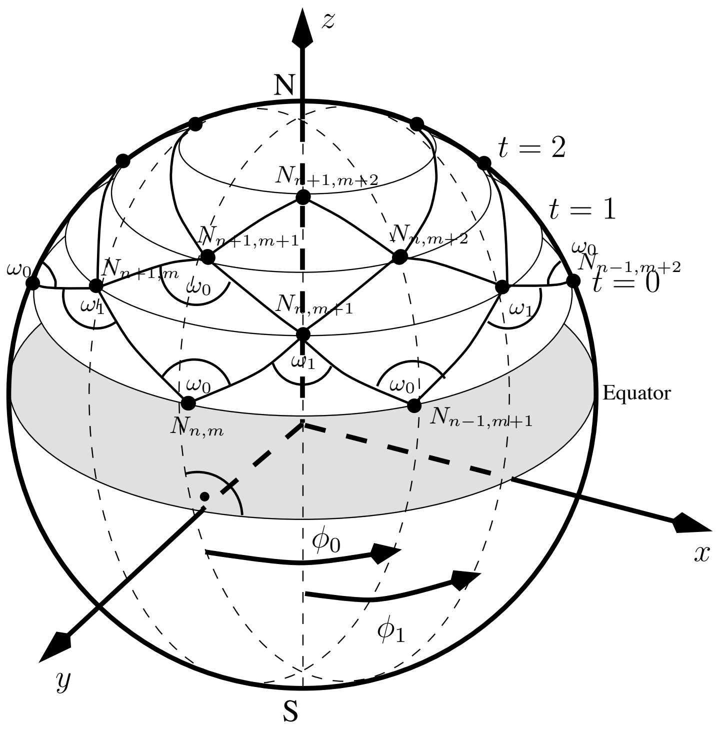
if in addition

$$N_{n+1,m+1} = 2\frac{\langle N_{n+1,m} + N_{n,m+1}, N_{n,m} \rangle}{\| N_{n+1,m} + N_{n,m+1} \|^2}(N_{n+1,m} + N_{n,m+1}) - N_{n,m} \qquad (4.2.33)$$

then $N$ is a lorentzharmonic weak Chebychev net in $S^2$

**Proposition 4.2.1** *(Bobenko,Pinkall [7]) Any lorentzharmonic weak Chebychev net in $S^2$ $N : \mathbb{Z} \to S^2 \subset \mathbb{R}^3$ is the Gauss map of a discrete K-surface, which is determined by $N$ uniquely up to homothety and translations.*

**Definition 4.2.2** *If $\delta_1 = \delta_2$ we call the above defined $N$ forms a lorentzharmonic discrete Chebychev net. Such a discrete Gauss map will be also called discrete $O^3$ Invariant Chiral Model (see also [44] and (1.1.2)).*




**Definition 4.2.3** *A rotational invariant lorentzharmonic discrete Chebychev net is a lorentzharmonic discrete Chebychev net with the properties:*

a.) *All points $N_{n,m}$ for a fixed initial time $t = n+m = 0$ lie on a circle henceforth called time circle.*

b.) *The angles in an initial Cauchy zig-zag $(...N_{n+1,m}, N_{n,m}, N_{n,m+1}, N_{n-1,m+1}...)$ are equal for a fixed time, i.e. if $\omega_{n,m}$ is the arc defined by the arcs*

$$\omega_{n,m} := (arc(N_{n+1,m}, N_{n,m}), arc(N_{n,m}, N_{n,m+1}))$$

*in a spherical square then $\omega_{n,m} \overset{Def}{=} \omega_{n-1,m+1} \overset{Def}{=} \omega_0$ for $n+m=0$ and if*

$$\omega'_{n,m+1} := (arc(N_{n,m}, N_{n,m+1}), arc(N_{n,m+1}, N_{n-1,m+1}))$$

*then $\omega'_{n,m+1} \overset{Def}{=} \omega'_{n+1,m} \overset{Def}{=} \omega_1$ for $n+m=0$.*

Let $d_{i,j}$ be a great circle which cuts the per assumption nondegenerate angles $\omega_0$ and $\omega_1$, respectively into halves at the point $N_{i,j}$.

Let us call the great circle, which is parallel to the time circle $t = 0$ the equator.

**Lemma 4.2.4** *All $d_{i,j}$ (defined as above are perpendicular to the time circles $t = const$ if the nondegenerate evolution takes place between the equator and the north pole (see Figure 1).*

**Proof:**
Let the evolution be as in (4.2) (see Figure 1). $d_{n,m+1}$ with $n+m=0$ is parallel to the height of the spherical isoceles, which is defined by extending the arcs between $N_{n,m}$ and $N_{n,m+1}$ and the arcs between $N_{n-1,m+1}$ and $N_{n,m+1}$ to the equator. Hence $d_{n,m+1}$ is perpendicular to $t = 0$.

Since all arcs of the initial cauchy zig-zag have the same lengths and $d_{n,m+1}$ is perpendicular to $t = 0$ the distance between $N_{n,m+1}$ and the time circle $t = 0$ is the same for all $N_{n,m+1}$ with $n+m=0$ . Hence $t = 1$ is parallel to $t = 0$.

Since $t = 1$ is parallel to $t = 0$ and the arcs $arc(N_{n+1,m}, N_{n+1,m+1})$ and $arc(N_{n+1,m+1}, N_{n,m+1})$ with $n+m=0$ have the same length , it follows analogously that $t = 2$ is parallel to $t = 1$. Note that $d_{n,m}$ is defining the diagonal in the spherical square $(N_{n,m}, N_{n+1,m}, N_{n+1,m+1}, N_{n,m+1})$ and that $d_{n,m}$ is parallel to the height of the isoceles defined by the extended arcs between $N_{n+1,m}$, $N_{n+1,m+1}$ and $N_{n,m+1}$. It is perpendicular to $t = 0, 1, 2$.

By induction it follows that the $d_{i,j}$ are perpendicular to all $t = const$.

Moreover it follows analogously that all points $N_{n+k,m+k}$ with $n+m=0$ lie on $d_{n,m}$.

If the evolution runs backwards, i.e. starts in a small circle around the north pole down to the equator the same arguments apply.

□

If the evolution crosses the equator or the north pole, respectively then the argumentation gets a little more subtle. In particular here sometimes degeneracies occur. In the following we want to exclude these cases and investigate only pieces of the surfaces which are defined on the above mentioned wellbehaving strip.

By Lemma (4.2.4) all $d_{i,j}$ meet in the north pole. Let $U(\phi_0)$ be a rotation around the pole axes, which rotates the great circle $d_{n,m}$ into $d_{n-1,m+1}$, $n+m=0$. Let $U(\phi_1)$ be a rotation around the pole axes, which rotates the great circle $d_{n,m+1}$ into $d_{n-1,m+2}$, $n+m=0$. Since $d_{n,m+1}$ cuts the time line arc between $N_{n,m}$ and $N_{n-1,m+1}$ into halves it follows that $U(\frac{\phi_0}{2})$ rotates $d_{n,m}$ into $d_{n,m+1}$ and finally that $U(\frac{\phi_0}{2})U(\frac{\phi_1}{2})=U(\phi_1)$. Hence the longitudinal angles $\phi_1$ and $\phi_0$ are the same.

By Lemma (4.2.4) and the fact that the $d_{i,j}$'s are the diagonals of the fundamental squares in our Chebychev net we see that for a rotational invariant Gauss map it suffices to know the evolution of $N$ along a fundamental strip defined by two consecutive great circles $d_{n+1,m}$ and $d_{n,m}$. By rotational symmetry we recover the rest by a rotation with $U(\phi_0)^k$.

Let us choose a coordinization on $\mathbb{R}^3$ such that $d_{n,m}=\begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}$. for fixed $n,m\in\mathbb{N}$. By the above we know that

$$N_{n+j,m+j}=\begin{pmatrix}0\\ N^2_{n+j,m+j}\\ N^3_{n+j,m+j}\end{pmatrix}\qquad j\in\mathbb{Z}.$$

In addition we know that the rotation around the z-Axes $U(\phi_0)$ turns the normals $N_{n+j+1,m+j}$ into $N_{n+j,m+j+1}$:

$$U(\phi_0):=\begin{pmatrix}\cos\phi_0 & -\sin\phi_0 & 0\\ \sin\phi_0 & \cos\phi_0 & 0\\ 0 & 0 & 1\end{pmatrix}\tag{4.2.34}$$

$$N_{n+j,m+j+1}=U(\phi_0)N_{n+j+1,m+j}\tag{4.2.35}$$

The same holds for the normals along the neighbouring great circle $d_{n,m}$:

$$N_{n+j-1,m+j+1}=U(\phi_0)N_{n+j,m+j}\tag{4.2.36}$$

Let us study the evolution along the above described strip. Inserting into (4.2.33) and using (4.2.35) and (4.2.36) we have to distinguish two equations:
Let

$$G:=U(\frac{\phi_0}{2})^*+U(\frac{\phi_0}{2})\tag{4.2.37}$$

$$\begin{aligned}N_{n+j+1,m+j+2} &= 2\frac{\langle N_{n+j+1,m+j+1}+N_{n+j,m+j+2},N_{n+j,m+j+1}\rangle}{\|N_{n+j+1,m+j+1}+N_{n+j,m+j+2}\|^2}(N_{n+j+1,m+j+1}\\ &\quad +N_{n+j,m+j+2})-N_{n+j,m+j+1}\\ &= 2\frac{\langle G\,U(\frac{\phi_0}{2})\,N_{n+j+1,m+j+1},N_{n+j,m+j+1}\rangle}{\|G\,U(\frac{\phi_0}{2})\,N_{n+j+1,m+j+1}\|^2}G\,U(\frac{\phi_0}{2})\,N_{n+j+1,m+j+1}\\ &\quad -N_{n+j,m+j+1}\end{aligned}\tag{4.2.38}$$

and

$$\begin{aligned}
N_{n+j+1,m+j+1} &= 2\frac{\langle N_{n+j+1,m+j}+N_{n+j,m+j+1}, N_{n+j,m+j}\rangle}{\|N_{n+j+1,m+j}+N_{n+j,m+j+1}\|^2}(N_{n+j+1,m+j} \\
&\quad +N_{n+j,m+j+1})-N_{n+j,m+j} \\
&= 2\frac{\langle G\,U(\frac{\phi_0}{2})^*\,N_{n+j,m+j+1}, N_{n+j,m+j}\rangle}{\|G\,U(\frac{\phi_0}{2})^*\,N_{n+j,m+j+1}\|^2}\,G\,U(\frac{\phi_0}{2})^*\,N_{n+j,m+j+1} \\
&\quad -N_{n+j,m+j} \qquad (4.2.39)
\end{aligned}$$

Note that

$$[G, U(\frac{\phi_0}{2})] = 0. \qquad (4.2.40)$$

Define

$$\begin{aligned}
q_{2j} &:= N_{n+j,m+j} & q_{2j+1} &:= U(\tfrac{\phi_0}{2})^*\,N_{n+j,m+j+1} \\
p_{2j} &:= U(\tfrac{\phi_0}{2})\,N_{n+j,m+j} & p_{2j+1} &:= N_{n+j,m+j+1}
\end{aligned}$$

Hence

$$q_{2j} = U(\frac{\phi_0}{2})^*\,p_{2j} \quad \text{and} \quad q_{2j+1} = U(\frac{\phi_0}{2})^*\,p_{2j+1} \qquad (4.2.41)$$

Inserting $p$ and $q$ into equations (4.2.38) and (4.2.39) we get:

$$\begin{aligned}
p_{2j+3} &= 2\frac{\langle G\,p_{2j+2}, p_{2j+1}\rangle}{\|G\,p_{2j+2}\|^2}G\,p_{2j+2}-p_{2j+1} \\
q_{2j+2} &= 2\frac{\langle G\,q_{2j+1}, q_{2j}\rangle}{\|G\,q_{2j+1}\|^2}G\,q_{2j+1}-q_{2j}
\end{aligned}$$

Using (4.2.41) and (4.2.40) we get:

$$\begin{aligned}
q_{2j+1} &= U(\frac{\phi_0}{2})^*\,p_{2j+1} \\
&= 2\frac{\langle G\,U(\frac{\phi_0}{2})\,q_{2j}, U(\frac{\phi_0}{2})\,q_{2j-1}\rangle}{\|G\,U(\frac{\phi_0}{2})\,q_{2j}\|^2}U(\frac{\phi_0}{2})^*\,G\,U(\frac{\phi_0}{2})\,q_{2j}-U(\frac{\phi_0}{2})^*\,p_{2j-1} \\
&= 2\frac{\langle G\,q_{2j}, q_{2j-1}\rangle}{\|G\,q_{2j}\|^2}G\,q_{2j}-q_{2j-1}
\end{aligned}$$

In the same way:

$$p_{2j} = U(\frac{\phi_0}{2})q_{2j} = 2\frac{\langle Gp_{2j-1}, p_{2j-2}\rangle}{\|Gp_{2j-1}\|^2}Gp_{2j-1}-q_{2j-2}.$$

As expected the evolution of $q$ along the circle $d_{n,m}$ and of $p$ along $d_{n,m+1}$ is exactly the same:

$$q_{k+1} = 2\frac{\langle G\,q_k, q_{k-1}\rangle}{\|G\,q_k\|^2}\,G\,q_k-q_{k-1} \qquad (4.2.42)$$

In the previously introduced euclidean coordinates on $\mathbb{R}^3$ $q_k$ can be parametrized as follows:

$$q_k = \begin{pmatrix} 0 \\ \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} \qquad \alpha \in (0, \frac{\pi}{2}). \tag{4.2.43}$$

Hence (see 4.2.34)

$$G\,q_k = 2\begin{pmatrix} \cos\phi/2 & 0 & 0 \\ 0 & \cos\phi/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} = \begin{pmatrix} 0 \\ 2\begin{pmatrix} \cos\phi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} \end{pmatrix}.$$

Define

$$C^{-1} := 2\begin{pmatrix} \cos\phi/2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad v_k = \begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} \in S^1$$

then the evolution of $v_k$ is given by

$$v_{k+1} = 2\frac{\langle C^{-1}\,v_k, v_{k-1}\rangle}{\|C^{-1}\,v_k\|^2}\,C^{-1}\,v_k - v_{k-1}. \tag{4.2.44}$$

The evolution defined by (4.2.44) is called a discrete Neumann system [41],[42]. If the Matrix $C^{-1}$ is not diagonal we call the evolution defined by (4.2.44) an extended discrete Neumann system.
Let $A := a_1 + ia_2$, $B := b_1 + ib_2$, $a_1, a_2, b_1, b_2, \alpha \in \mathbb{R}$. The equation

$$e^{i\alpha_{k+1}+i\alpha_{k-1}} = \frac{A\,e^{i\alpha_k} + Be^{-i\alpha_k}}{\bar{A}\,e^{-i\alpha_k} + \bar{B}e^{i\alpha_k}} \tag{4.2.45}$$

is called the squareroot of a discrete pendulum equation and appears also inthe theory of billards (cf. last section). [6].

**Theorem 4.2.5**
*An evolution sequence $(e^{i\alpha_k})_{k\in\mathbb{N}}$ belonging to the evolution defined by the square-root of the discrete pendulum equation with $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and the initial angles* $e^{\alpha_0}, e^{\alpha_1}$ $\Leftrightarrow$
*to the sequence*

$$\left(\begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix}\right)_{k\in\mathbb{N}}$$

*belonging to the evolution defined by the extended discrete Neumann system with*

$$C^{-1} := \begin{pmatrix} a_1 + b_1 & b_2 - a_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \quad \textit{and initial vectors} \quad \begin{pmatrix} \cos\alpha_0 \\ \sin\alpha_0 \end{pmatrix}, \begin{pmatrix} \cos\alpha_1 \\ \sin\alpha_1 \end{pmatrix}.$$

**Proof:**

"⇒"

$$e^{i\alpha_{k+1}} + e^{i\alpha_{k-1}} = \frac{A\,e^{i\alpha_k} + Be^{-i\alpha_k}}{\bar{A}\,e^{-i\alpha_k} + \bar{B}e^{i\alpha_k}}\,e^{-i\alpha_{k-1}} + e^{i\alpha_{k-1}} \qquad (4.2.46)$$
$$2\frac{Re[(A\,e^{i\alpha_k} + B\,e^{-i\alpha_k})e^{-i\alpha_{k-1}}]}{\|A\,e^{i\alpha_k} + B\,e^{-i\alpha_k}\|^2}\,(A\,e^{i\alpha_k} + B\,e^{-i\alpha_k})$$

Now if

$$z_k = x_k + iy_k \quad \text{and} \quad z_k^R = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$

then $\langle w_1^R, w_2^R\rangle_{R^2} = Re(w_1\bar{w}_2)$. Furthermore

$$\begin{aligned} z = A\,e^{i\alpha_k} + B\,e^{-i\alpha_k} &= (a_1+b_1)\cos\alpha_k + i(a_2+b_2)\cos\alpha_k \\ &\quad + i(a_1-b_1)\sin\alpha_k - (a_2-b_2)\sin\alpha_k \end{aligned} \qquad (4.2.47)$$

⇒

$$z^R = \begin{pmatrix} a_1+b_1 & b_2-a_2 \\ a_2+b_2 & a_1-b_1 \end{pmatrix}\begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix} \qquad (4.2.48)$$

Inserting (4.2.47),(4.2.48) into (4.2.47) finishes the proof.

"⇐" The complexified extended discrete Neumann system reads as:

$$e^{i\alpha_{k+1}} + e^{i\alpha_{k-1}} = \nu_k(A\,e^{i\alpha_k} + B\,e^{-i\alpha_k}) \qquad \nu_k \in \mathbb{R}$$

⇒

$$\frac{e^{i\alpha_{k+1}} + e^{i\alpha_{k-1}}}{e^{-i\alpha_{k+1}} + e^{-i\alpha_{k-1}}} = \frac{A\,e^{i\alpha_k} + Be^{-i\alpha_k}}{\bar{A}\,e^{-i\alpha_k} + \bar{B}e^{i\alpha_k}} = e^{i\alpha_{k+1}+i\alpha_{k-1}}$$

□

In our application $C^{-1}$ (4.2.44) is diagonal and we find that the evolution equation for the height $\alpha$ of a rotational invariant surface (see (4.2.43)) is given by

$$e^{i\alpha_{k+1}+i\alpha_{k-1}} = \frac{e^{2i\alpha_k} + k}{1 + ke^{2i\alpha_k}} \qquad (4.2.49)$$

with $k = -\frac{1-cos\frac{\phi}{2}}{1+cos\frac{\phi}{2}} = -tan^2\frac{\phi}{4}$

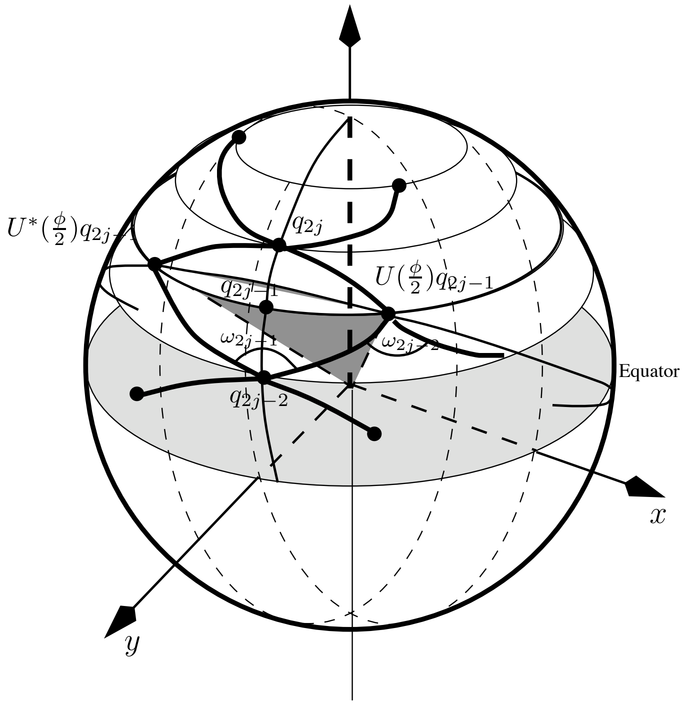
**Proposition 4.2.6** *The double of the height $\alpha$ of a rotational invariant surface defined as in (4.2.43) satisfies the doubly discrete pendulum equation*

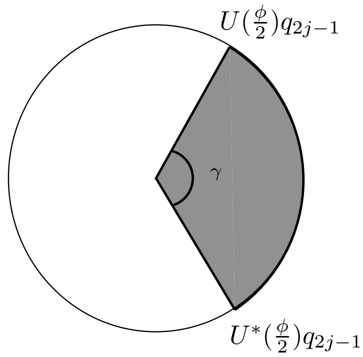
$$q_{k+1}q_{k-1} = \left(\frac{q_k + k}{1 + kq_k}\right)^2$$

*for $k = -tan^2\frac{\phi}{4}$, $q_k = e^{2i\alpha_k}$.*

Since we know that the angle $\omega$ between the edges of a fundamental face of the rotational invariant Chebychev net on the two-sphere satisfies as well as the above $\alpha$ the doubly discrete pendulum equation, just for in general different kinds of $k$ (see e.g. (1.2.13)) it would be interesting to find a connection between these two angles.

This would be in particular interesting in view of a possible quantization of the $O^3$ invariant Chiral model [44]. Let us consider a fundamental face $(q_{2j-2}, U^*(\frac{\phi}{2})q_{2j-1},\, q_{2j}, U(\frac{\phi}{2})q_{2j-1})$ of the Chebychev net on the two-sphere:




Let $\gamma$ be the length of the diagonal $(U(\frac{\phi}{2})q_{2j-1}, U^*(\frac{\phi}{2})q_{2j-1})$ in our fundamental square, $\delta$ be the length of the edges of the square and $\omega_{2j-1}$ the angle between the edges as indicated in the figure above.
Then

$$2\sin\frac{\gamma}{2} = |U(\frac{\phi}{2})\begin{pmatrix} 0 \\ \cos\alpha_{2j-1} \\ \sin\alpha_{2j-1} \end{pmatrix} - U^*(\frac{\phi}{2})\begin{pmatrix} 0 \\ \cos\alpha_{2j-1} \\ \sin\alpha_{2j-1} \end{pmatrix}| = 2\sin\frac{\phi}{2}\cos\alpha_{2j-1}$$

Now on the other hand by the formula of Delambré [8]:

$$\sin\frac{\gamma}{2} = \sin\frac{\omega_{2j-1}}{2}\sin\delta.$$

Hence

$$\cos\alpha_{2j-1} = \underbrace{\frac{\sin\delta}{\sin\phi/2}}_{a}\sin\frac{\omega_{2j-1}}{2}; \qquad \frac{\phi}{2} \neq 0 \tag{4.2.50}$$

After some simple trigonometric manipulations we finally obtain for the case $\frac{\omega}{2}, \alpha \in [0, \pi), a \geq 0$

$$e^{-i\alpha} = a \sin\frac{\omega}{2} + \sqrt{a^2 \sin^2\frac{\omega}{2} - 1} \tag{4.2.51}$$

If we consider a rotational invariant surface belonging to a solution of the initial value problem $(\omega_0, \omega_1)$:

$$e^{i\omega_{k+1}+i\omega_{k-1}} = \left(\frac{k + e^{i\omega_k}}{1 + ke^{i\omega_k}}\right)^2 \qquad k = tan^2\frac{\delta}{2} \tag{4.2.52}$$

where $\omega$ gives the angles between the normals of the corresponding Gaussmap (1.2.13) then scanning through all possible values of $\delta$ changes basically only the size of the lattice constants of our discrete surface and not the general features of the surface.
So let $\delta$ be fixed but arbitrary. Then the choice of the constant $a$ in (4.2.50)

$$\sin\frac{\phi}{2} = \frac{1}{a}\sin\delta \qquad \phi, \delta \in (0, \pi)$$

amounts to fixing the rotation angle $\phi$. But if we fix the angles $\phi$ and $\delta$ then the initial value $\omega_0$ determines the initial value $\omega_1$ and hence finally the solution to the initial value problem. In this sense the parameter $a$ classifies rotational invariant surfaces.
Let us consider an example. Let $a = 1$, i.e. $\frac{\phi}{2} = \delta$, then by (4.2.51) we find

$$e^{i\alpha_k} = ie^{i\omega_k/2} \tag{4.2.53}$$

Inserting (4.2.53) into (4.2.49) with $k = -tan^2\frac{\phi}{4}$ we find immediately that

$$e^{i\frac{\omega_{k+1}}{2}+i\frac{\omega_{k-1}}{2}} = \frac{\tilde{k} + e^{i\omega_k}}{1 + \tilde{k}e^{i\omega_k}} \qquad \tilde{k} = tan^2\frac{\delta}{2} = -k$$

which finally gives (4.2.52) as expected. The surface corresponding to $a = 1$ should be the pseudosphere, which can be seen by the fact that $\frac{\phi}{2} = \delta$ implies $\omega_1 = \pi$ along the equator. The above shall serve as a start of a thorough investigation of the Gauss map of discrete K-surfaces.

## 4.3 The last sine-Gordon type equation

Next to the discussed examples of sine-Gordon equations (see preceding chapters) there exists another nice model where a "nearly" sine-Gordon equation finds its application. It will turn out that the rotational invariant Chebychev net case of last sections is a consequence of this model.

Consider the sphere $S^2 \in \mathbb{R}^3$. We choose an arbitrary initial Cauchy zig-zag on the sphere whose "lower" points lie all on one circle. This circle shall serve as latitudinal meridian and hence defines a north pole which shall per definition lie in time up direction. The points on the sphere which are the vertices of the above initial Cauchy zig-zag shall evolve according to the rules:

1.) The point $N_u$ which closes the spherical quadrilateral formed by the "zig": $N_l, N_d, N_r$ shall lie on the same longitudinal meridian as $N_d$.

2.) The lengths of the arcs $N_l, N_u$ and $N_u, N_r$ are subject to the constraint:

$$\langle N_r, N_d\rangle\langle N_l, N_d\rangle = \langle N_r, N_u\rangle\langle N_l, N_u\rangle \tag{4.3.54}$$

where $\langle , \rangle$ denotes the euclidean product on $\mathbb{R}^3$.

If we exclude the case $N_d = N_u$ then the above rules define a unique evolution on $S^2$, as will be shown below.

**Proposition 4.3.1** a.) *The latitudes of the points on $S^2 \in \mathbb{R}^3$ which obey the above defined evolution satisfy a square root of the discrete sine-Gordon equation.*

b.) *Any solution to the discrete sine-Gordon equation with the above intial conditions and appropriate choice of root defines an evolution of the above form.*

**Proof:**
a.) and b.) can be proven simultanously by showing the equivalence of (4.3.54) to a discrete sine-Gordon equation.

The constraint in (4.3.54) is independent of the choice of reference frame in $SO(3)$ hence we can choose a reference frame such that

$$N_d = \begin{pmatrix} 0 \\ \cos(\alpha_d) \\ \sin(\alpha_d) \end{pmatrix} \quad \text{and} \quad N_u = \begin{pmatrix} 0 \\ \cos(\alpha_u) \\ \sin(\alpha_u) \end{pmatrix}.$$

Moreoever if we denote the distances of $N_r$ or $N_l$ to the diagonal $arc(N_u, N_d)$ with $\phi_r$ or $\phi_l$ respectively we obtain:

$$N_l = \begin{pmatrix} -\sin(\phi_l)\cos(\alpha_l) \\ \cos(\phi_l)\sin(\alpha_l) \\ \sin(\alpha_l) \end{pmatrix} \quad \text{and} \quad N_r = \begin{pmatrix} \sin(\phi_l)\cos(\alpha_r) \\ \cos(\phi_r)\sin(\alpha_r) \\ \sin(\alpha_r) \end{pmatrix}.$$

A straightforward computation shows that writing condition (4.3.54) in these coordinates gives the following sine-Gordon type equation:

$$e^{i\alpha_u + i\alpha_d} = \frac{k_r + e^{2i\alpha_r}}{1 + k_r e^{2i\alpha_r}} \frac{k_l + e^{2i\alpha_l}}{1 + k_l e^{2i\alpha_l}} \tag{4.3.55}$$

where

$$k_i = \frac{\cos\left(\frac{\phi_i}{2}\right) - 1}{\cos\left(\frac{\phi_i}{2}\right) + 1}.$$

We call this equation of sine-Gordon type since the evolution for the variables $k_r, k_l$ will in general be highly nontrivial and not constant, as assumed for the ordinary discrete sine-Gordon equation.

Clearly in the case $k_r = k_l = k$ and $\alpha_r = \alpha_l$ we are back at the case of a rotational invariant Chebychev net (4.2.3).

In general one can't expect that the above evolution defines a discrete Chebychev net. In fact we conjecture that the above evolution of points on the sphere forms a discrete (weak) Chebychev net only for the rotational invariant case.

# Chapter 5

# Appendix

## 5.1 Poisson relations for the free edge algebra

Let us recall the classical commutation relations of the $2p$ space periodic algebra generated by variables, which are

assigned to the faces of discrete Minkowski spacetime, i.e. the socalled face algebra $\{(p_k)_{k\in\{0,...2p-1\}}\}$

$$\begin{aligned} \{p_k, p_{k+1}\} &= 8a & (5.1.1) \\ \{p_n, p_{k+1+j}\} &= 0 \qquad \text{for} \quad j \in \{1...2p-3\} & (5.1.2) \\ p_k &= p_{2p+k}, & (5.1.3) \end{aligned}$$

(where usually $a = -\frac{1}{4}$). The periodic face variables $\{(p_k)_{k\in\{0,...2p-1\}}\}$ shall be expressed in terms of variables on the edges of discrete Minkowski space time - the socalled edge variables $\{(u_k, v_k)_{k\in\{0,...2p\}}\}$ - via :

$$p_k = u_k + u_{k+1} - v_k + v_{k+1} + \beta \qquad \beta = const, \tag{5.1.4}$$

(where $\beta$ is usually zero). The edge variables $\{(u_k, v_k)_{k\in\{0,...2p+1\}}\}$ shall be free coordinates on a virtual phase space $\mathbb{R}^{4p+4}$, i. e. there shall be a priori no restrictions as e.g. the one given in section 2.4.2. In the following we would like to construct poisson relations between the edge variables $\{(u_k, v_k)_{k\in\{0,...2p+1\}}\}$ which are partially induced by the poisson relations between the face variables via (5.1.4) and a few additional assumptions. Regarding (5.1.4)we see that the generators of the edge algebra do not necessarily need to be periodic, in order to induce that the generators of the face algebra $\{(p_k)_{k\in\{0,...2p-1\}}\}$are. So we will introduce translational invariant monodromies $m_{u\,even/odd}$ and $m_{v\,even/odd}$ (see also (5.1.6)). The algebra generated by $\{(u_k, v_k)_{k\in\{0,...2p+1\}}\}$ will henceforth be quasiperiodic (see also section 2.3). Let us impose the following assumptions:

*Assumptions:*

1.) *The commutation relations shall be translational invariant, i.e.:*

$$\{a_k, a_j\} = \{a_{k+m}, a_{j+m}\} \qquad f.\ a. \quad m \in \mathbb{Z} \quad a = u, v. \tag{5.1.5}$$

2.) *The monodromies defined by:*

$$\begin{array}{lllllll} m_{u\,even} & := & u_{2p} - u_0 & m_{u\,odd} & := & u_{1+2p} - u_1 & \\ m_{v\,even} & := & v_{2p} - v_0 & m_{v\,odd} & := & v_{1+2p} - v_1 & n \in \mathbb{Z} \end{array} \tag{5.1.6}$$

*shall define quasi periodic variables recursively via:*

$$\begin{array}{lllllll} u_{2n+2p} & := & u_{2n} + m_{u\,even} & u_{2n+1+2p} & := & u_{2n+1} + m_{u\,odd} & \\ v_{2n+2p} & := & v_{2n} + m_{v\,even} & v_{2n+1+2p} & := & v_{2n+1} + m_{v\,odd} & n \in \mathbb{Z} \end{array} \tag{5.1.7}$$

3.) $u_i$ *and* $v_j$ *on different sites* $i, j$ *within the periodicity regime shall poisson commute,i.e:*

$$\{u_i, v_{i+k}\} = 0 \qquad f.\ a. \quad k \in \mathbb{Z} - \{0\} mod/, 2p \qquad i \in \mathbb{Z}. \tag{5.1.8}$$

4.) $u_i$ *and* $v_j$ *shall observe the following ultralocality condition:*

$$\{a_i, c_{i+k}\} = 0 \qquad f.\ a. \quad k \in \mathbb{Z} - \{0, -1\}\, mod\, 2p \quad a = u, v \quad i \in \mathbb{Z} \tag{5.1.9}$$

5.) *The poisson structure on quasi periodic phase space shall be compatible with the poisson structure given by (5.1.1)-(5.1.3) via definition (5.1.4). In particular we assume that the poisson bracket between any two coordinates is a real constant function.*

(5.1.10)

By using (5.1.3) we notice now the following:

$$\begin{aligned} 0 = p_{2n+2p} - p_{2n} &= m_{u\,even} + m_{u\,odd} - m_{v\,even} + m_{v\,odd} \\ 0 = p_{2n+1+2p} - p_{2n+1} &= m_{u\,even} + m_{u\,odd} + m_{v\,even} - m_{v\,odd} \end{aligned}$$

Hence $m_{u\,even} = -m_{u\,odd} := m_u$ and $m_{v\,even} = m_{v\,odd} := m_v$, i.e. modulo the minus sign even or odd monodromies of the $u_i$'s and $v_i$'s have to be the same, respectively. Hence we can restrict ourselves to the investigation of $4p+2$ variables $\{(u_k, v_k)_{k\in\{0,\dots 2p\}}\}$

**Proposition 5.1.1** *The poisson structure for the quasi periodic variables* $\{(u_k, v_k)_{k\in\{0,\dots 2p\}}\}$ *or* $\{(u_k, v_k)_{k\in\{0,\dots 2p-1\},m_u,m_v}\}$ *is up to the constants* $b, c \in \mathbb{R}$ *uniquely determined* *by axioms (5.1.5)-(5.1.10), it reads as:*

$$
\begin{aligned}
\{u_n, u_{n+2l}\} &= -2a - b + c \qquad l \in \{1...p-1\} \\
\{u_n, u_{n+2m-1}\} &= 2a + b - c \qquad m \in \{1...p\} \\
\{v_n, v_{n+k}\} &= 2a - b - c \qquad k \in \{1...2p-1\} \\
\{u_n, v_{n+j}\} &= 0 \qquad j \in \mathbb{Z} - \{0\}\, mod\, 2p \\
\{u_n, v_n\} &= -2a - c \qquad n \in \mathbb{Z} \\
\{u_n, m_u\} &= (-1)^{n+1} 4a + 2b - 2c \\
\{u_n, m_v\} &= 0 \\
\{v_n, m_u\} &= 0 \\
\{v_n, m_v\} &= 4a - 2b - 2c
\end{aligned}
\tag{5.1.11}
$$

**Proof:**
Because of (5.1.10) define $\{u_i, v_i\} = -2a - c$ and $\{u_n, u_{n+1}\} = 2a + b - c$ with $b, c \in \mathbb{R}$. Now

$$
\begin{aligned}
8a = \{p_n, p_{n+1}\} &\overset{5.1.4}{=} \{u_n + u_{n+1} - v_n + v_{n+1}, u_{n+1} + u_{n+2} - v_{n+1} + v_{n+2}\} \\
&\overset{5.1.8,5.1.9}{=} \{u_{n+1} + v_{n+1}, u_{n+1} + u_{n+2} - v_{n+1} + v_{n+2}\} \\
&= \{v_{n+1}, v_{n+2}\} + 6a + b + c
\end{aligned}
$$

By (5.1.8) we get

$$
\begin{aligned}
0 &\overset{5.1.9}{=} \{u_{n+1}, c_{n+2}\} = \{u_{n+1}, u_{n+2}\} + \{u_{n+1}, u_{n+3}\} \Rightarrow \{u_{n+1}, u_{n+3}\} = -2a - b + c \\
0 &\overset{5.1.9}{=} \{v_{n+1}, c_{n+2}\} = -\{v_{n+1}, v_{n+2}\} + \{v_{n+1}, v_{n+3}\} \Rightarrow \{v_{n+1}, v_{n+3}\} = 2a - b - c.
\end{aligned}
$$

Hence by induction

$$
\begin{aligned}
\{u_n, u_{n+2l}\} &= -2a - b + c \qquad l \in \{1...p-1\} \\
\{u_n, u_{n+2m-1}\} &= 2a + b - c \qquad m \in \{1...p\} \\
\{v_n, v_{n+k}\} &= 2a - b - c \qquad k \in \{1...2p-1\}
\end{aligned}
\tag{5.1.12}
$$

Now

$$
\begin{aligned}
-2a - b + c &= \{u_{2n}, u_{2n+2l}\} = \{u_{2n}, u_{2n+2l-2p} + m_u\} \qquad l \in \{1...p-1\} \\
&= -\{u_{2j}, u_{2j+2p-2l}\} + \{u_{2n}, m_u\} \qquad j = n + l - p \\
&= 2a + b - c + \{u_{2n}, m_u\}
\end{aligned}
$$

Analogously we obtain $\{u_{2n-1}, m_u\} = 4a + 2b - 2c$ where again $2p$ denotes periodicity.

Now

$$\begin{aligned}\{u_n, p_{n+2p-1}\} &= \{u_n, p_{n-1}\} = -4a - b \\ &\overset{5.1.4}{=} -2a - b + c + \{u_{n+2p}, v_{n+2p}\}\end{aligned}$$

Hence $\{u_{n+2p}, v_{n+2p}\} = -2a - c$.
Analogously we find all other poisson commutation relations between coordinates $u_i, v_i$ with the monodromies as well as between the monodromies themselves. After the check wether all in this way obtained commutation relations are compatible with the poisson structure given by (5.1.1)-(5.1.3) via definition (5.1.4) we are done.

**Remark 5.1.2** *The monodromies commute with the face variables for all commutation relations given in (5.1.11.*

□

# Conclusions

The investigation of classical phase space belonging to the discrete sine-Gordon model resulted in an explanation of the relation between the vertex variables $g_i$ and the difference variables or face variables $p_i = g_{i-1} - g_{i+1}$. The difference variables can be obtained via a reduction of phase space [45, 26]. The reduction process deleted the additional degrees of freedom one obtaines by integrating the difference variables $p_i$ along a Cauchy zig-zag, i.e.

$$g_{2l} = g_0 - \sum_{2k+1}^{l-1} p_{2k+1}. \tag{5.1.13}$$

A similar proces takes place in the nonabelian case, when integrating the frame of a discrete K-surface, i.e.

$$\phi_{t,k} = L_{t,k} \dots L_{0,0} \phi_{0,0} \tag{5.1.14}$$

So it will be of particular interest to investigate how the techniques developped for the above abelian case (5.1.13) carry over to the nonabelian case (5.1.14). This could lead to a possibility to quantize K-surfaces, or at least the normal map of K-surfaces, which is still an open problem.

Another way to achieve this goal could be to understand better how a quantized K-surface or its correponding quantized normal map should look like. The connection of the rotational invariant normal map to the pendulum equation (which posesses a meaningful quantization) give a valuable hint for that.

Another important question is wether the normal map of a K-surface could may be fit into another - more general - integrable model as there are e.g. the wellknown Toda systems [28], which already posess a quantized analog [34].

Besides this "futuristic" outlook there are more down to earth open questions. In the quantization of the models described in the present work appears for the root of unity case the phenomen of classical background [52]. In [52] it was already indicated how this feature could be interpreted within the realms of noncommutative geometry [12]. Nevertheless this has to be investigated further. In particular it will be here espiacially interesting, wether the lightcone shifts introduced in the present work (3.4.55) already posses a formulation within this theory. This applies also the the theory of quantum groups ([25]).

# Bibliography


[1] B. Feigin A. Antonov, *Quantum group representations and baxter equation*, hep-th/9603105 (1996).

[2] G. Barnich, M. Henneaux, and C. Schomblond, *Covariant description of the canonical formalism*, Phys. Rev. D **44** (1991), no. 4, R939–R941.

[3] J. Bellisard, Operator Algebras and Application (Cambridge) (M. Takesaki D. E. Evans, ed.), vol. II, Cambridge University Press, Cambridge, 1988.

[4] H. Bethe, *Zur theorie der metalle; i. eigenwerte und eigenfunktionen der linearen atomkette*, Z. d. Phys. **71** (1931), 205–226.

[5] A. Bobenko, *Surfaces in terms of 2 by 2 matrices. old and new integrable cases*, Harmonic Maps and Integrable Systems (A. Fordy and J. Woods, eds.), Vieweg, 1994.

[6] A. Bobenko, N. Kutz, and U. Pinkall, *The discrete quantum pendulum*, Physical Letters A **177** (1993), 399–404.

[7] A. Bobenko and U. Pinkall, *Discrete surfaces with constant negative gaussian curvature and the hirota equation*, SFB 288 Preprint Nr. 127, 1994.

[8] I. N. Bronstein and K. A. Semendjajew, Taschenbuch der Mathematik, Verlag Harri Deutsch/ Interdruck Leipzig, GDR, 1981.

[9] M. Bruschi, O. Ragnisco, P. M. Santini, and G.-Z. Tu, *Integrable symplectic maps*, Physica D **49** (1991), 273–294.

[10] H.W. Capel and F. W. Nijhoff, *Integrable quantum mappings*, Centre de Recherches Mathématiques, CRM Proceedings and Lecture Notes, 1994.

[11] S. Coleman, *Quantum sine-gordon equation as the massive thirring model*, Phys. Rev. D **11** (1975), no. 8, 2088–2097.

[12] A. Connes, *Geometrie non commutative*, InterEditions, 1990.

[13] C. Crnkovic and E. Witten, Covariant description of canonical formalism in geometrical theories, in Three hundred years of gravitation, 676–6844, Cambridge University Press, 1987, pp. 676–6844.

[14] C. Destri and H.J. de Vega, Light-Cone Lattice Approach to Fermionic Theories in 2D *the massive thirring model*, Nucl. Phys. B **290** (1987), 363–391.

[15] R. G. Douglas, *Banach algebra techniques in operator theory*, Pure and Applied Mathematics, vol. 49, Academic Press N.Y., 1972.

[16] C. Emmrich and N. Kutz, *Doubly discrete lagrangian systems related to the hirota and the sine-gordon equation*, Phys. Lett. **A 201** (1995), 156–160.

[17] L. D. Faddeev, *The bethe ansatz*, Sfb 288 Preprint Nr. 70, available at http://www-sfb288.math.tu-berlin.de, 1993.

[18] L. D. Faddeev and R. M. Kashaev, *Generalized bethe ansatz equations for the hofstadter problem*, Comm. Math. Phys. **169** (1995), 181–191.

[19] L. D. Faddeev and L. A. Takhtajan, *Hamilton methods in the theory of solitons*, Springer Series in Soviet Mathematics, 1987.

[20] L. D. Faddeev and A. Yu. Volkov, *Quantum inverse scattering method on a spacetime lattice*, Theor. Math. Phys. **92** (1992), 837–842.

[21] ______, *Abelian current algebra and the virasora algebra on the lattice*, Phys. Lett. B **315** (1993), 311.

[22] ______, *Hirota equation as an example of an integrable symplectic map*, Lett. Math. Phys. **32** (1994), 125–136.

[23] V.A. Fateev and A. B. Zamolodchikov, *The exactly solvable case of a 2d lattice of plane rotators*, Phys. Lett. A **92** (1982), no. 1, 35–39.

[24] Bjorn Felsager, *Geometry, particles and fields*, second ed., Odense University Press, 1981.

[25] J. Fuchs, *Affine lie algebras and quantum groups*, Cambridge University Press, 1992.

[26] V. Guillemin and S. Sternberg, Symplectic techniques in physics, Cambridge University Press, 1984.

[27] R. Haag, *Local quantum physics*, Springer, Texts and Monographs in Physics, 1992.

[28] R. Hirota, Nonlinear Partial Difference Equations II*: Discretetime toda equation*, J. Phys. Soc. Japan **43** (1977), no. 6, 2074–2078.

[29] ______, Nonlinear Partial Difference Equations III *: Discrete sine-gordon equation*, J.Phys.Soc.Japan **43** (1977), no. 6, 2079–2086.

[30] T. Hoffmann, *Diskrete k-flaechen*, Master's thesis, TU Berlin, 1995.

[31] D. Hofstadter, *Energy levels and wave functions of bloch electrons in rational and irrational magnetic fields*, Phys. Rev. B **14** (1976), 2239–2249.

[32] A.G. Izergin and V.E. Korepin, *The lattice quantum sine-gordon model*, Lett. Math. Phys. **5** (1981), 199–205.

[33] R. Seiler J. Bellisard, Ch. Kreft, *Analysis of the spectrum of a particle on a triangular lattice with two magnetic fluxes by algebraic and numerical methods*, J. Phys. A **24** (1991), 2329–2353, Math. General.

[34] R. M. Kashaev and N. Reshetikhin, *Affine toda field theory as a 3 dimensional integrable system*, hep-th/9507065 (1995).

[35] A.N. Kirillov and N. Yu. Reshetikhin, *Exact solution of the integrable xxz heisenberg model with arbitrary spin: I. the ground state and the excitation spectrum*, J. Phys. A: Math. Gen. **20** (1987), 1565–1585.

[36] T.R. Klassen and E. Melzer, *Sine-gordon $\neq$ massive thirring, and related heresies*, hep-th 9206114 (1992).

[37] Ch. Kreft and R. Seiler, *Models of the hofstadter type*, Sfb 288 Preprint (1996), no. 209.

[38] N. Kutz, *On the spectrum of the quantum pendulum*, Phys. Lett. A **187** (1994), 365–372.

[39] ______, *Free massive fermions inside the quantum discrete sine-gordon model*, Sfb 288 Preprint (1996).

[40] L.Bianchi, Vorlesungen "Uber Differentialgeometrie, B. G. Teubner, Leipzig, first ed., 1899.

[41] J. Moser and A. P. Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Comm. Math. Phys. **139** (1991), 217–243.

[42] C. Neumann, *De problemato quodam mechanico quod ad primam integralium ultraellipticorum classem revocatur*, J. Reine Angew. Math. **56** (1859), 46–63.

[43] F.W. Nijhoff, H.W. Capel, and V. G. Papageorgiu, *Integrable quantum mappings*, Phys. Rev. A **46** (1992), no. 4, 2155–2158.

[44] K. Pohlmeyer, *Integrable hamiltonian systems and interactions through quadratic constraints*, Comm. Math. Phys. **46** (1976), 207–221.

[45] J. E. Marsden R. Abraham, Foundations of Mechanics, Benjamin/Cummings, Menlo Park, second ed., 1978.

[46] E.K. Sklyanin, L.A. Takhtadzhan, and L.D. Faddeev, *Quantum inverse problem method i*, Theor. Math. Phys. **40** (1979), no. 2, 194–220.

[47] F.A. Smirnov, *Connection between the sine-gordon model and the massive bose thirring model*, Theor. Math. Phys. **53** (1982), 323–334.

[48] Yu. B. Suris, *Integrable mappings of the standardtype*, Funct. Anal. App. **23** (1989), 74–79.

[49] Yu.B. Suris, *Generalized toda chains in discrete time*, Leningrad.Math.J. **2** (1991), no. 2, 339–352.

[50] L.A. Takhtadzhan and L.D. Faddeev, *The quantum method of the inverse problem and the heisenberg xxz model*, Russ. Math. Surveys **34** (1979), no. 5, 11–68.

[51] V.O. Tarasov, L.A. Takhtadzhyan, and L.D. Faddeev, *Local hamiltonians for integrable quantum models on a lattice*, Theor.Math.Phys. **57** (1983), no. 2, 163–181.

[52] N. Reshetikhin V. Bazhanov, A. Bobenko, *Quantum discrete sine-gordon model at roots of 1: Integrable quantum system on the integrable classical background*, Comm. Math. Phys. **175** (1996), 377–400.

[53] A.P. Veselov, *Integrable maps*, Russ. Math. Surv. **46** (1991), no. 5, 3–45, In Russian.

[54] A.Yu. Volkov, *Quantum volterra model*, Phys. Lett. A **167** (1992), 345–355.

[55] P. B. Wiegmann and A. V. Zabrodin, *Quantum group and magnetic translations, bethe-ansatz for the asbel-hofstadter problem.*, Nucl. Phys. B **422** (1994), 495–514.

[56] N. Woodhouse, *Geometric quantization*, Oxford University Press, 1980.

[57] W. Wunderlich, *Zur differenzengeometrie der flächen konstanter negativer krümmung*, Sitzungsber. Ak. Wiss. **160** (1951), 39–77.

[58] G. Zuckerman, Action Principles and global geometry, mathematical aspects of string theory, 1986.