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Spectra of Quantum Integrals

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Introduction

This article is about the spectral theory of Schrödinger operators which are used to describe electric particles moving in discrete 2-dimensional space under the influence of a perpendicular magnetic field. It is a wonderful and fruitful coincidence that among these operators are some which are invariant under automorphisms describing time evolutions of integrable doubly discrete quantum field theories. So this article is also about the spectral theory of quantum integrals, a name which we give to integrals of motions of such automorphisms. This adds a different point of view to the investigation of discrete models in solid state physics and makes the whole machinery of the inverse scattering method, in particular the Bethe ansatz, available for the task to determine the spectrum of the corresponding Schrödinger operators.

The operators which are investigated here are generalizations of the Hofstadter Hamiltonian which is the famous Hamiltonian describing an elementary Quantum-Hall system; this is a discretized version of an electron in a constant magnetic field, see [**?**]. The aim is to analyse their spectra. Early numerical work of Hofstadter [**?**] indicated a rich fractal structure in what is nowadays called the Hofstadter Butterfly. This points out the difficulty of the task: one commonly expects that for values of the magnetic flux which are irrational (in the appropriate units) the spectra of Hamiltonians of Hofstadter type are Cantor sets (nowhere dense sets without isolated points). A lot of work has been devoted to study of their spectra using various methods including semiclassical analysis [**?**, **?**, **?**], C^*-algebra and K-theory, functional analysis, and measure theory. These methods provide information on topological properties of the spectrum, such as rigorous proofs that, for certain irrational values of the flux, the spectrum of the Hofstadter Hamiltonian is a Cantor set – we mention a few works [**?**, **?**, **?**, **?**] refering the reader to [**?**] and [**?**] for further references – or continuity of band edges in the flux variable [**?**, **?**]. They also provide information on the Lebesgues measures of the spectra [**?**, **?**, **?**] and on the measure theoretic nature of the spectral measures. The last point is connected to the representation in which the operators act, for an overview on results regarding that question for the almost Mathieu operator see [**?**]. Specific spectral values for irrational flux were, however, not obtained (apart from the value 0 in the Hofstadter spectrum).

The methods which will be employed here are different from the above. They come under the name Bethe ansatz, and we shall concentrate our analysis on two of its variants: the polynomial (or functional) Bethe ansatz as formulated by Wiegmann et al. in [?] and the algebraic Bethe ansatz which was adapted to the present case by one of the authors [?]. We will present here a simplified approach to the polynomial Bethe ansatz which is based on a representation theoretic property, namely the existence of polynomial solutions in a representation of the Weyl-Heisenberg algebra. This approach, in which the possible rôle of quantum groups is de-emphasised, has the advantage that it makes the neccessary structural ingredients clear and shows how both works, that of [?] and that of [?], are interrelated. In fact, polynomial solutions for the family of operators in question can be obtained in two ways, one leading to the Bethe ansatz-equations and eigenvalues of [?] and the other containing the eigenvalues of [?] in a special case ($k = 1$). The two solutions have a fundamental difference. Whereas the Bethe ansatz-equations and eigenvalues which were obtained in [?] do not show any regular behaviour under a small change of the magnetic field – in fact, they can only be formulated for rational flux – does the other solution give rise to eigenvalues and -functions which depend analytically on the strength of the magnetic field. In particular, like all other solutions which may be obtained by the algebraic Bethe ansatz, the latter solutions exist for irrational flux, too.

Apart from Wiegmann et al. [?] also Faddeev and Kashaev have approached the calculation of exact spectral values using a generalized form of the Bethe-ansatz [?]. We will not discuss their work here. It contains the Bethe ansatz equations of [?] as special cases, and it is also restricted to rational flux.

We do only find a finite number of eigenvalues for the operators. It is therefore natural to ask whether they play a special rôle in the spectrum. In a first attempt to give an answer to this question we are led to the conjecture that for the quantum integral related to the quantum pendulum, the solutions obtained by the Bethe ansatz describe the touching of bands in the spectrum for rational flux.

The article is intended partly as a review article, including an understandable overview, though neccessarily partial, on the used techniques and models with particular emphasis on their discrete aspects. The first section is about the discrete Weyl-Heisenberg algebra and some of its representations including the irreducible ones for rational flux. This algebra arises in different fields of mathematics under different headings. In algebra it is called the rotation algebra and in geometry the quantized torus. Some of its representations have to be employed to compute the spectra for rational flux. This is discussed in Section 2 together with the simplification which arrises if a Chambers relation holds. In Section 3 we present the family of operators which are to be investigated. We show that they are invariant under certain automorphisms of the Weyl-Heisenberg algebra, an observation which is important for Sections 5 und 6. Section 4 is devoted to a technique for computing eigenvalues and eigenfunctions of the operators which may be seen as the essence of the polynomial Bethe ansatz. We discuss two qualitative different ways to obtain polynomial eigenfunctions and specify the result to two operators of particular interest: the Hofstadter Hamiltonian and the QP-integral. The mathematics used in Section 4 is comparatively simple but can only be applied to a small subclass of operators. In order to go behond this subclass one has to use a more sophisticated method which has its origin in integrable quantum field theory. We discuss in Section 5 how to express the operators of interest as quantum integrals of discrete sine Gordon field theory. The sixth section is then devoted to the adaptation of the algebraic Bethe ansatz to the present situation. It also contains the determination of where in the bands, which exist for rational flux, the eigenvalues found by the Bethe ansatz lie. A comparison between the results obtained by the polynomial Bethe ansatz and by the algebraic Bethe ansatz concludes the article.

Before coming to the main part of the article let us devote a paragraph to the motivation of one of our two main themes: discrete Schrödinger operators describing the motion of electric particles in $\mathbb{Z}^2$ under the influence of a perpendicularly applied magnetic field.

The discrete form of the Landau Hamiltonian

Landau considered the quantum mechanical problem of a particle with electric charge 1 and mass 1 in the plane $\mathbb{R}^2$ (spanned by e_1, e_2) to which a constant magnetic field of magnitude γ along e_3 is applied. The particle is not subject to any other potential so that its Hamiltonian, acting on $L^2(\mathbb{R}^2)$, is [**?**]

$$H_L = \frac{v_1^2 + v_2^2}{2}$$

where $v_j = p_j - A_j$ is the velocity operator (in units $c = \hbar = 1$) which contains a vector potential A for the magnetic field, i.e. $\partial_1 A_2 - \partial_2 A_1 = \gamma$. $v_1^2 + v_2^2$ may be termed magnetic Laplacian, because it generalizes the usual (2-dimensional) Laplacian which is obtained for $\gamma = 0$. However one chooses the gauge, one obtains the commutation relation

$$[v_1, v_2] = i\gamma. \tag{0.1}$$

This commutation relation is algebraically the same as Heisenberg's commutation relation. In fact, the latter may be obtained upon identification of v_1 with the position operator, v_2 with the momentum operator, and $2\pi\gamma$ with Planck's constant. Under this identification H_L becomes algebraically the harmonic oscillator and hence has discrete, equally spaced, unbounded spectrum which is bounded from below. Due to the above algebraic similarity we call the exponentiated velocity operators

$$W_\gamma(a) = \exp -i(a_1 v_1 + a_2 v_2) \tag{0.2}$$

also Weyl-operators. Commutation relation (**??**) implies for the Weyl-operators

$$W_\gamma(a) W_\gamma(b) = e^{-\frac{i\gamma}{2}\sigma(a,b)} W_\gamma(a+b) \tag{0.3}$$

where $\sigma(a,b) = a_1 b_2 - a_2 b_1$ is the canonical symplectic form on $\mathbb{R}^2$. To describe the action of $W_\gamma(a)$ on $L^2(\mathbb{R}^2)$ we choose the symmetric gauge, that is take $A_1 = -\frac{\gamma}{2}x_2$, $A_2 = \frac{\gamma}{2}x_1$. In that gauge

$$W_\gamma(a)\ \Psi(x) = e^{-\frac{i\gamma}{2}\sigma(a,x)}\Psi(x-a)\ . \tag{0.4}$$

From that equation follows that, for all $a, b \in \mathbb{R}^2$

$$[W_{-\gamma}(a), W_\gamma(b)] = 0.$$

In particular, $W_{-\gamma}(a)$ commutes with the Landau Hamiltonian H_L and hence the family $\{W_{-\gamma}(a)\}_{a\in\mathbb{R}^2}$ plays the rôle of a symmetry. Only if $\gamma = 0 \pmod{2\pi}$ this family forms a group, namely the translation group. In general, the product is twisted by a phase (compare (**??**)) and hence $\{W_{-\gamma}(a)\}_{a\in\mathbb{R}^2}$ furnish a projective representation of the translation group. For that reason these operators have been named magnetic translations [**?**]. To repeat, we have two families of Weyl-operators, $\{W_\gamma(a)\}_{a\in\mathbb{R}^2}$ and $\{W_{-\gamma}(a)\}_{a\in\mathbb{R}^2}$. Algebraically they differ only by a sign in their product but their rôle for the Landau model is different. The generators of the first family are the velocities and give rise to the Hamiltonian, whereas the second family furnishes the symmetry operators. Moreover, it can be proved [**?**] that the set of bounded operators on $L^2(\mathbb{R}^2)$ which commute with all $W_\gamma(a)$, $a \in \mathbb{R}^2$, is equal to the closure in the weak operator topology of the set $\{W_{-\gamma}(a)\}_{a\in\mathbb{R}^2}$. Hence the two families are in a sense dual to each other. Differing algebraically only by the value for the magnetic field, the $W_\gamma(a)$ are also called magnetic translations although they should, strictly speaking, better be called dual magnetic translations. The generators of the $W_{-\gamma}(a)$ are, of course, $k = p + A$. They have commutation relation $[k_1, k_2] = -i\gamma$ and give rise to the integrals of motion. One usually chooses here the Landau centre c, $c_1 = k_2\gamma^{-1}$, $c_2 = -k_1\gamma^{-1}$. This is the centre of gyration of the particle in the constant magnetic field.

What happends if we discretize space, that is, if we allow the particle to move only on $\mathbb{Z}^2 \subset \mathbb{R}^2$? In that case, in order to obtain operators acting on the natural Hilbert space of the discrete setting $\ell^2(\mathbb{Z}^2)$, the generators v_j, $j = 1, 2$, have to be exponentiated along one lattice spacing the length of which we chose to be 1. According to one possible scheme of how to replace differentials by differences one then obtains that the discrete magnetic Laplacian, which acts on $\ell^2(\mathbb{Z}^2)$, is given by

$$(1-W_\gamma(1,0)^*)(1-W_\gamma(1,0))+(1-W_\gamma(0,1)^*)(1-W_\gamma(0,1)) =: 4-H_{Hof}. \tag{0.5}$$

$H_{Hof} = W_\gamma(1,0)+W_\gamma(1,0)^*+W_\gamma(0,1)+W_\gamma(0,1)^*$ is the well known Hofstadter Hamiltonian. The Hofstadter Hamiltonian thus plays, up to a constant, the rôle of the discrete Landau Hamiltonian. We are not only interested in this Hamiltonian but also in its generalizations involving translations to the next nearest neighbours. It is clear that H_{Hof} belongs to the closure in the norm topology of the algebra which is (even linearly) generated by the set $\{W_\gamma(n)|n \in \mathbb{Z}^2\}$. This algebra shall be called discrete Weyl-Heisenberg algebra. Its representation theory lies at the heart of this work.

1 Discrete Weyl-Heisenberg algebra

The discrete Weyl-Heisenberg algebra is at the root of the following analysis. It comes also under the name of rotation algebra and of quantized torus. The discrete Weyl-Heisenberg algebra is the basic element of the algebra of observables of the discrete sine-Gordon model and in particular of the quantum pendulum.

In this first section we define again the discrete Weyl-Heisenberg algebra but this time abstractly. We discuss some of its motivations and go through a list of representations which will be used in the latter part of this article. In particular, we will construct a faithful family of irreducible representations of the Weyl-Heisenberg algebra.

1.1 Definition of the discrete Weyl-Heisenberg algebra $\mathcal{A}_\gamma$

We have introduced above the discrete Weyl-Heisenberg algebra in terms of Weyl-operators acting on $\ell^2(\mathbb{Z}^2)$. As often in mathematics, it is usefull to abstractly characterize the structure formed by the Weyl-operators. This is the C^*-algebra based on the relation $W_\gamma(1,0)W_\gamma(0,1) = e^{-i\gamma}W_\gamma(0,1)W_\gamma(1,0)$.

Let $\gamma \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, $q = e^{i\gamma}$, and $\mathcal{A}_\gamma^0$ be the $*$-algebra over the field of complex numbers generated by two elements u, v (and their image under the $*$-operation u^*, v^*) modulo the ideal generated by the relations

$$uu^* - 1, \quad u^*u - 1, \quad vv^* - 1, \quad v^*v - 1, \quad uv - q^{-1}vu.$$

This means that any element x of $\mathcal{A}_\gamma^0$ may be represented by a finite sum

$$x = \sum c_n u^{n_1} v^{n_2}, \qquad (n \in \mathbb{Z}^2, c_n \in \mathbb{C}).$$

In fact, by the first four relations, the elements represented by u and v are invertible, and their inverses are equal to u^* and v^*, respectively. The last relation allows one to order monomials in such a way that all u's stand to the left of v's.

On $\mathcal{A}_\gamma^0$ we define a norm as follows: Consider all representations of $\mathcal{A}_\gamma^0$ by bounded operators on some Hilbert space, that is all $*$-homomorphisms $\rho : \mathcal{A}_\gamma^0 \to \mathcal{B}(\mathcal{H})$. Then, for $x \in \mathcal{A}_\gamma^0$, define

$$\|x\| := \sup\{\|\rho(x)\| |\rho \text{ a representation of } \mathcal{A}_\gamma^0\} \tag{1.1}$$

and $\mathcal{A}_\gamma$ to be the completion of $\mathcal{A}_\gamma^0$ with respect to this norm. This is by definition the discrete Weyl-Heisberg algebra. By construction $\mathcal{A}_\gamma$ is a C^*-algebra and equation (**??**) defines its unique norm [**?**].

$\mathcal{A}_\gamma$ is also called the rotation algebra for reasons which will become clear below. Furthermore, it may as well be understood as the twisted group algebra of $\mathbb{Z}^2$ with twisting element $e^{-\frac{i\gamma}{2}\sigma}$ where $\sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ is given by $\sigma(n,m) = n_1m_2 - m_1n_2$. This means essentially that $\mathcal{A}_\gamma^0$ has a basis given by

$$W(n) := e^{i\frac{\gamma}{2}n_1n_2}u^{n_1}v^{n_2}, \tag{1.2}$$

$n \in \mathbb{Z}^2$, and the product of these basis elements is the twisted product

$$W(n)W(m) = e^{-\frac{i\gamma}{2}\sigma(n,m)}W(n+m). \tag{1.3}$$

Any representation of $\mathcal{A}_\gamma$ gives then rise to a projective representation of $\mathbb{Z}^2$. We shall see below that the $W(n)$ coincide in a representation on $\ell^2(\mathbb{Z}^2)$ with the Weyl-operators $W_\gamma(n)$ introduced in the last section.

We shall have to consider the angle γ as parameter and consider the dependence of certain properties of elements of $\mathcal{A}_\gamma$ on this parameter. This may appear a priori a bit artifical, since some important algebraic properties of $\mathcal{A}_\gamma$ are strongly dependent on the value of γ. But already the interpretation of γ as a physically measurable quantity (the flux through a unit cell) indicates that certain features should depend in a regular way on γ. The technical context which is needed here is that of the family of all $\mathcal{A}_\gamma$ forming a continuous, differentiable or even analytic field of C^*-algebras. But for our purposes it will be sufficient to view the operators as elements which depend parametrically on γ, the dependence showing up in the way how u and v commute.

Now we list two basic properties of the Weyl-Heisenberg algebra. The second one will be a consequence of Theorem **??** below.

- If $\frac{\gamma}{2\pi}$ is irrational then $\mathcal{A}_\gamma$ is a simple C^*-algebra, in particular all its non-trivial representations are faithful (injective).
- In contrast, if $\frac{\gamma}{2\pi} = \frac{M}{N}$ and M, N coprime, $\mathcal{A}_\gamma$ has a large centre. It is generated by u^N and v^N; therefore it is isomorphic to $C(\mathrm{T}^{\mathbf{2}})$. It follows that all elements of the form $u^N - z$ or $v^N - z$, $z \in \mathbb{C}$ being of modulus 1, generate ideals of $\mathcal{A}_\gamma$.

We shall have to consider a variety of representations of the discrete Weyl-Heisenberg algebra which we usually denote by $(\rho, \mathcal{H})$. But in order to avoid cumbersome notation we shall adopt the convention to write for the action of $u \in \mathcal{A}_\gamma$ on a vector ψ of a representation space $\mathcal{H}$ simply $u \cdot \psi$, the $*$-homomorphism ρ between $\mathcal{A}_\gamma$ and $\mathcal{B}(\mathcal{H})$ being understood.

1.2 Discrete quantum mechanics: the Weyl-Schrödinger representation

In this subsection, we define a family of representations of the discrete Weyl-Heisenberg algebra, which is of particular importance. It is the analoge of the famous Schrödinger representation in the continuous case.

For any given angle θ there is a representation of $\mathcal{A}_\gamma$ on $L^2(S^1)$, defined by

$$\begin{aligned} u \cdot f(z) &= e^{i\theta} f(q^{-1}z) \\ v \cdot f(z) &= z f(z). \end{aligned} \tag{1.4}$$

These representations are discrete integrated forms of the Schrödinger representation on S^1 of the Heisenberg commutation relation $2\pi[q,p] = -ih$. In fact, v acts in this representation as the exponentiated position operator and u as translation by γ (for $\theta = 0$). We shall therefore call these representations Weyl-Schrödinger representations. These representations will be of particular use for the polynomial (functional) Bethe ansatz (see Section **??**).

The $*$-algebra generated by the representative of v is that of Laurent polynomials in one complex variable, and hence its norm-closure may be identified with the continuous functions on the circle $C(S^1)$. Under this identification, conjugating a function $f \in C(S^1)$ by u results in the same as the automorphism on $C(S^1)$ induced by the rotation of S^1 about the angle γ. This is the origin of the name rotation algebra.

Often the Fouriertransforms of the Weyl-Schrödinger representations are considered. Under Fouriertransform $L^2(S^1) \to \ell^2(\mathbb{Z})$: $f \mapsto \psi$,

$$\psi(n) := \frac{1}{2\pi i} \oint z^{-n-1} f(z) dz$$

the action becomes

$$\begin{aligned} u \cdot \psi(n) &= e^{i\theta} q^{-n} \psi(n) \\ v \cdot \psi(n) &= \psi(n-1). \end{aligned} \tag{1.5}$$

The famous Almost Mathieu equation is the eigenvalue equation for the operator $u + u^* + k(v + v^*)$, $k \in \mathbb{R}$, $k > 0$, in such a representation.

To end this subsection, let us summarize some of the properties of the discrete Schrödinger representation (see also subsection 1.4).

- If $\frac{\gamma}{2\pi}$ is irrational, the Weyl-Schrödinger representations are irreducible. In other words, the von Neumann closure of such a representation is $\mathcal{B}(\mathcal{H})$ (it is a type I factor). For different $\theta\,'s$ modulo 2π, the representations are inequivalent.
- If $\frac{\gamma}{2\pi}$ is rational, the representations are reducible. Moreover, these representations are not faithful.
- The family of all Weyl-Schrödinger representations, parametrized by the angle θ, is faithful in the sense that only $0 \in \mathcal{A}_\gamma$ acts in all representations as the 0-operator.

1.3 Discrete magnetic translations

Consider the representation on $\ell^2(\mathbb{Z}^2)$ given by

$$\begin{aligned} u \cdot \psi(n) &= e^{iA_1(n)} \psi(n_1 - 1, n_2) \\ v \cdot \psi(n) &= e^{iA_2(n)} \psi(n_1, n_2 - 1) \end{aligned} \tag{1.6}$$

where $A : \mathbb{Z}^2 \to \mathbb{R}^2$ is a gauge potential with discrete rotation γ:

$$(A_2(n) - A_2(n_1 - 1, n_2)) - (A_1(n) - A_1(n_1, n_2 - 1)) = \gamma.$$

The function A for given magnetic flux γ is unique up to a discrete gradient. For different gauges the representations are unitarily equivalent. If we choose the symmetric gauge $A_1(n) = -\frac{\gamma}{2}n_2$, $A_2(n) = \frac{\gamma}{2}n_1$ then the operators $W(n)$ defined in (**??**) act as

$$W(n) \cdot \psi(k) = e^{-\frac{i\gamma}{2}\sigma(n,k)}\psi(k-n) \tag{1.7}$$

and hence coincide with the Weyl operators $W_\gamma(n)$ defined in the introduction. Thus they generate the algebra of observables for an electric particle moving in $\mathbb{Z}^2$ to which a homogenuous magnetic field is perpendicularly applied. The angle γ is proportional to the magnetic field or the flux per unit cell. For simplicity we call $\frac{\gamma}{2\pi}$ the flux. Let us include a proof of the following well known fact:

Theorem 1 *The representation of the discrete Weyl-Heisenberg algebra $\mathcal{A}_\gamma$ by discrete magnetic translations is faithful.*

Proof: The positive linear functional $\tau : \mathcal{A}^0_\gamma \to \mathbb{C}$, $\tau(W(n)) = \delta_0(n)$, δ_k denoting the function on $\mathbb{Z}^2$ which is 1 on k and 0 elsewhere, extends to a faithful normalized trace on $\mathcal{A}_\gamma$. Hence the GNS-representation with respect to τ is faithful. The Hilbertspace of that representation is the completion of $\mathcal{A}_\gamma$ with respect to the scalar product $(x, y) := \tau(x^*y)$. Hence we may take the $W(n)$, $n \in \mathbb{Z}^2$ as a basis of that space and the identification of $W(n)$ with the function δ_n defines a unitary operator between the GNS-representation space and $\ell^2(\mathbb{Z}^2)$. Taking into account that the action in the GNS-representation is given by left multiplication one straightforwardly checks that this unitary furnishes a unitary equivalence between the GNS-representation and representation (**??**). Thus faithfulness of the above representation follows from faithfulness of the GNS-representation and the fact that a change of gauge may be incorporated in another unitary equivalence. q.e.d.

The above shows that the representations on $\ell^2(\mathbb{Z}^2)$ may serve as a more concrete definition of $\mathcal{A}_\gamma$; we have already made use of this in the introduction. The representations are, however, reducible for all γ. In fact, for irrational $\frac{\gamma}{2\pi}$, their von Neumann closures are type II_1 factors. Like in the continuous case, in the symmetric gauge, the commutant of the representation is the von Neumann closure of the corresponding representation of $\mathcal{A}_{-\gamma}$ [**?**].

We have already mentionned the analogy between the magnetic flux γ and Planck's constant in quantum mechanics. This analogy leads to quantization of arbitrary functions on the space $\mathbb{Z}^2$ which plays the role of configuration space in the discrete Landau model and of phase space for discrete quantum mechanics. Let f be an element of the Schwartz test function space over $\mathbb{Z}^2$. Its quantization is naturally defined by

$$W(f) = \sum_{n\in\mathbb{Z}^2} f(n)\, W(n). \tag{1.8}$$

If γ were 0 (mod 2π) then the $W(f)$ would all commute and therefore generate the algebra of smooth functions over the torus ($W(f)$ is then the Fouriertransform of f). For general values of γ the discrete Weyl-Heisenberg algebra is therefore also named – in the spirit of Connes' non commutative geometry – quantized (or non-commutative) torus. The Fouriertransform $\hat{f}$ of f is called the classical symbol of the operator $W(f)$.

1.4 Irreducible representations

To construct a faithful family of irreducible representations of the discrete Weyl-Heisenberg algebra $\mathcal{A}_\gamma$ we decompose its faithful representation by magnetic translations into irreducible constituents. This decomposition is well known but we shall carry it out explicitly for the convenience of the reader. It leads to the following theorem about, what we shall call, standard irreducible representations:

Theorem 2

i) For irrational flux, the Weyl-Schrödinger representation of the discrete Weyl-Heisenberg algebra is irreducible. For rational flux this is not so.
ii) For rational flux, $\frac{\gamma}{2\pi} = \frac{M}{N}$*,* M *and* N *coprime, the irreducible representations of the discrete Weyl-Heisenberg algebra are labelled by elements of the two dimensional torus* $S \times S$*. Every one is unitarily equivalent to the standard representation defined below for some* $\vec{\theta}$ *in* $S \times S$*. The standard representations are for* u

$$u_{\vec{\theta}} = e^{\frac{i\theta_1}{N}} \begin{pmatrix} q^{-1} & & & & \\ & \ddots & & & \\ & & q^{-l} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{1.9}$$

and for v

$$v_{\vec{\theta}} = e^{\frac{i\theta_2}{N}} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}. \tag{1.10}$$

The labels $\vec{\theta}$ *will be called Bloch parameters.*

Proof: We prove first part of the theorem using the well known fact, that the rotation by an irrational angle on S^1 is ergodic. Stated differently, any set which is invariant under the rotation and has nonzero Lebesgues measure must coincide with the whole S^1 (up to a zero-measure set). Now suppose that the Weyl-Schrödinger representation were reducible and let P be the projection onto one of the proper invariant subspaces under the action of $\mathcal{A}_\gamma$. It follows that P commutes with the action of u. Since the von Neumann closure of the $*$-algebra generated by u (and its inverse) in this representation is all of $L^\infty(S^1)$, and this algebra is maximal commutative, P must be an element of it. Hence P is a characteristic function on a measurable set. Since P commutes as well with the action of v this measurable set must be invariant under rotation with γ. Hence for irrational $\frac{\gamma}{2\pi}$, by ergodicity, P is either 0 or 1 which is a contradiction.

The second statement is a corrolary of Proposition **??** which will be proved at the end of this subsection. So we defer the proof of this statement until then. q.e.d.

In order to make the analysis simple we choose the Landau-gauge which is given by $A_1(n) = -n_2\gamma$, $A_2(n) = 0$. In that case

$$\begin{aligned} u \cdot \psi(n) &= q^{-n_2}\psi(n_1 - 1, n_2) \\ v \cdot \psi(n) &= \psi(n_1, n_2 - 1). \end{aligned} \tag{1.11}$$

This choice of a gauge suggests a particular way to decompose the representation into Weyl-Schrödinger representations in Fourierspace. Consider the unitary operator

$$\mathcal{F}_1 : \ell^2(\mathbb{Z}^2) \to \int_{S^1}^{\oplus} \frac{d\theta}{2\pi} \mathcal{H}_\theta .$$

$\mathcal{H}_\theta$ is for each θ a copy of $\ell^2(\mathbb{Z})$,[1] and the component $\hat{\psi}_\theta$ of $\mathcal{F}_1(\psi)$ in $\mathcal{H}_\theta$ is given by

$$\hat{\psi}_\theta(n_2) = \sum_{n_1 \in \mathbb{Z}} e^{in_1\theta}\psi(n).$$

Hence $\mathcal{F}_1$ maps the Hilbert space $\ell^2(\mathbb{Z}^2)$ into the Hilbert space of square integrable sections in the trivial bundle $S^1 \times \ell^2(\mathbb{Z})$. In this representation the action is

$$u \cdot \hat{\psi}_\theta(n_2) = q^{-n_2} \sum_{n_1 \in \mathbb{Z}} e^{in_1\theta}\psi(n_1 - 1, n_2) = e^{i\theta} q^{-n_2} \hat{\psi}_\theta(n_2)$$

and

$$v \cdot \hat{\psi}_\theta(n_2) = \hat{\psi}_\theta(n_2 - 1).$$

Hence u and v preserve the fibres $\mathcal{H}_\theta$. Summarizing we have shown

Proposition 1 *The representation (**??**) of $\mathcal{A}_\gamma$ on $\ell^2(\mathbb{Z}^2)$ decomposes into a direct integral of Weyl-Schrödinger representations parametrized by θ in S^1.*

We have argued before, that Weyl-Schrödinger representations for irrational fluxes are irreducible due to ergodicity of the corresponding circle action. Thus

[1] An element ϕ of $\int_{S^1}^{\oplus} \frac{d\theta}{2\pi}\mathcal{H}_\theta$ is a direct integral of elements of $\mathcal{H}_\theta$. Denoting the component of ϕ in $\mathcal{H}_\theta$ by ϕ_θ the scalar product is given by

$$< \phi, \phi > = \int_{S^1} \frac{d\theta}{2\pi} < \phi_\theta, \phi_\theta >_{\ell^2(\mathbb{Z})} .$$

for irrational flux, the decomposition into irreducible representations is completed. Let us therefore now suppose that $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M, N) = 1$ (which means that M and N coprime), and consider the unitary operator

$$\mathcal{F}_2 : \mathcal{H}_\theta \to \int_{S^1}^{\oplus} \frac{d\varphi}{2\pi} \mathcal{H}_{(\theta,\varphi)}.$$

For each φ, $\mathcal{H}_{(\theta,\varphi)}$ is a copy of $\mathbb{C}^N$.[2] The component $\hat{\phi}_\varphi$ of $\mathcal{F}_2(\phi)$ in $\mathcal{H}_{(\theta,\varphi)}$ is given by

$$\hat{\phi}_\varphi(l) = \sum_{n\in\mathbb{Z}} e^{in\varphi} \phi(nN + l), \tag{1.12}$$

$l \in \{1, \ldots, N\}$. Then, since $q^N = 1$,

$$u \cdot \hat{\phi}_\varphi(l) = e^{i\theta} q^{-l} \hat{\phi}_\varphi(l) \tag{1.13}$$

and

$$v \cdot \hat{\phi}_\varphi(l) = \begin{cases} e^{i\varphi} \hat{\phi}_\varphi(N) & \text{if } l = 1 \\ \hat{\phi}_\varphi(l-1) & \text{if } 2 \leq l \leq N \end{cases}. \tag{1.14}$$

Hence u and v preserve once again the fibres $\mathcal{H}_{(\theta,\varphi)}$.

Proposition 2 *For $\frac{\gamma}{2\pi} = \frac{M}{N}, (M, N) = 1$, the representation (**??**) of $\mathcal{A}_\gamma$ on $\ell^2(\mathbb{Z}^2)$ decomposes into a direct integral of N-dimensional irreducible representations parametrized over $S^1 \times S^1$. Acting on $\mathcal{H}_{(\theta,\varphi)}$ u has matrix representation*

$$u_\theta = e^{i\theta} \begin{pmatrix} q^{-1} & & & & \\ & \ddots & & & \\ & & q^{-l} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{1.15}$$

and v

$$v_\varphi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ e^{i\varphi} & 0 & \cdots & & 0 \end{pmatrix}. \tag{1.16}$$

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[2] Denoting the component of $\phi \in \int_{S^1}^{\oplus} \frac{d\varphi}{2\pi} \mathcal{H}_{(\theta,\varphi)}$ in $\mathcal{H}_{(\theta,\varphi)}$ by ϕ_φ the scalar product is given by

$$< \phi, \phi >= \int_{S^1} \frac{d\varphi}{2\pi} \sum_{l=1}^{N} \overline{\phi_\varphi(l)} \phi_\varphi(l).$$

Two such matrix representations, one for (θ,φ) and one for (θ',φ'), are unitarily equivalent if and only if $\theta-\theta'=n\gamma$ for some $n\in\mathbb{Z}$, and $\varphi=\varphi'$, that is if $u_\theta^N=u_{\theta'}^N$ and $v_\varphi^N=v_{\varphi'}^N$.

Proof: The matrix representations may be directly read off from (**??**,**??**). Since these two matrices generate the algebra of all complex valued $N\times N$ matrices the representations are irreducible. If $\theta'=\theta-n\gamma$ then $u_{\theta'}=v_\varphi^{-n}u_\theta v_\varphi^n$, and hence $u_{\theta'}$ and u_θ are unitarily equivalent. On the other hand, since $u_\theta^N=e^{iN\theta}\mathrm{id}$ and $v_\varphi^N=e^{i\varphi}\mathrm{id}$ the conditions $\theta'=\theta-n\gamma$ for $n\in\mathbb{Z}$ and $\phi'=\phi$ are also neccessary to insure that the representation labelled by (θ,ϕ) and that labelled by (θ',ϕ') are unitarily equivalent. q.e.d.

In the last proposition it looks like there was an asymmetry in the parameters labelling the irreducible components. This is of course not the case. In fact, it is easely seen that the irreducible representation in the proposition labelled by (θ,φ) is unitarily equivalent to the standard irreducible representation with parameters $\vec{\theta}=(\theta_1=N\theta,\theta_2=\varphi)$ given by (**??**,**??**). Since the represention on $\ell^2(\mathbb{Z}^2)$ is faithful the above discussed matrix representations furnish a faithful family of representations. That these representations are in fact all irreducible representations up to unitary equivalence, as is claimed in Theorem **??**, may be seen as follows. The map $\imath:\mathcal{A}_{\frac{2\pi M}{N}}\to\mathrm{Mat}_N(C(S^1\times S^1))$, $\imath(u)(\theta,\varphi)=u_\theta$, $\imath(v)(\theta,\varphi)=v_\varphi$, u_θ and v_φ as in (**??**,**??**), extends to an injective $*$-homomorphism of C^*-algebras. If ρ is an irreducible representation of $\mathcal{A}_{\frac{2\pi M}{N}}$, it induces an irreducible representation of $\mathrm{Mat}_N(C(S^1\times S^1))\cong\mathrm{Mat}_N(\mathbb{C})\otimes C(S^1\times S^1)$. The (non-trivial) irreducible representations of the latter algebra are those for which the first factor in the tensor product, $\mathrm{Mat}_N(\mathbb{C})$, is represented irreducibly on $\mathbb{C}^N$, and the second, $C(S^1\times S^1)$, is an evaluation representation $f\mapsto f(\theta,\varphi)$ for fixed $(\theta,\varphi)\in S^1\times S^1$. Since all irreducible representations of $\mathrm{Mat}_N(\mathbb{C})$ are unitary equivalent, ρ is unitary equivalent to representation (**??**,**??**) with that pair (θ,φ).

2 Computing Spectra for Rational Flux

We have seen that, for rational $\frac{\gamma}{2\pi}$, $\mathcal{A}_\gamma$ has finite dimensional representations. In a finite dimensional representation an operator is a matrix and its spectrum is the set of roots of the characteristic polynomial of this matrix. These roots may be computed numerically, or even analytically if the degree is not too high or in the presence of additional symmetries.

If $\{(\rho_i,\mathcal{H}_i)\}_{i\in I}$ is a faithful family of representations of $\mathcal{A}_\gamma$, i.e. if $\bigcap_{i\in I}\ker\rho_i=\{0\}$, then the spectrum $\sigma(H)$ of an element $H\in\mathcal{A}_\gamma$ is given by

$$\sigma(H)=\bigcup_{i\in I}\sigma(\rho_i(H)).$$

Thus, for rational $\frac{\gamma}{2\pi}$ we may take the set of all standard representations to obtain the spectrum $\sigma(H)$. Denoting by $H_{\vec{\theta}}$ the Hamiltonian in the standard representation with Bloch paramaters $\vec{\theta}$, $\sigma(H_{\vec{\theta}})$ is the set of roots of the characteristic polynomials $\det(H_{\vec{\theta}} - \lambda)$. $\sigma(H)$ is in general an infinite set, but for selfadjoint H, where it is a compact subset of $\mathbb{R}$, it consists of at most N bands (or partly points) and may therefore be computed numerically. However, since the size of the matrix $H_{\vec{\theta}}$ depends rather irregularly on $\frac{\gamma}{2\pi}$ this method is in principle useless to obtain $\sigma(H)$ for irrational $\frac{\gamma}{2\pi}$ even at an approximate level. Nevertheless, rather strong statements about continuity properties of $\sigma(H)$ in γ are known in cases. E.g., the band edges of $\sigma(H)$ vary continuously in γ [**?**, **?**]. For that reason, a picture which contains not only $\sigma(H)$ for fixed (rational) $\frac{\gamma}{2\pi}$ but for a whole family of them (e.g. for all $\frac{\gamma}{2\pi} = \frac{M}{N}$ with $N \le N_0$) can give an impression of what $\sigma(H)$ for irrational $\frac{\gamma}{2\pi}$ could be. Such a picture is often called a butterfly due to its aestethic appearence.

2.1 Chambers relation

The computation (as well as the analytical understanding) of $\sigma(H)$ simplifies enormously if H satifies a Chambers relation. The Chambers relation says something about the form of the dependence of the characteristic polynomial of $H_{\vec{\theta}}$ on $\vec{\theta}$. But it is not only a statement about H as an operator of $\mathcal{A}_\gamma$ with fixed γ but a statement in which γ enters as a parameter.

Definition 1 *An element $H \in \mathcal{A}_\gamma$, viewed to depend on γ through the relations in $\mathcal{A}_\gamma$, satisfies a Chambers relation, if for all rational $\frac{\gamma}{2\pi}$ there is a function $h_\gamma : S^1 \times S^1 \to \mathbb{C}$ and a polynomial p_γ such that*

$$\det(H_{\vec{\theta}} - \lambda) = p_\gamma(\lambda) + h_\gamma(\vec{\theta}). \tag{2.1}$$

The function h_γ is called off-set function.

Clearly, h_γ is continuous in $\vec{\theta}$, since $H_{\vec{\theta}}$ is, but its dependence on γ is not required to satisfy any regularity conditions; it is usually not continuous. Nevertheless, its dependence on γ is often less irregular as one might expect having the irregular dependence of p_γ on γ in mind. In fact, it will turn out that the off-set functions for the operators which are investigated here coincide up to a rescaling with the classical symbol of the operator.

To determine the spectrum of a selfadjoint element H satisfying a Chambers relation with off-set function h one only has to know of that function its maximum and minimum h_{max} and h_{min}, because $E \in \sigma(H)$ whenever $h_{min} \le -p(E) \le h_{max}$. Since the characteristic polynomials of $H_{\vec{\theta}}$ cannot have complex roots, and we may choose p and h to be real, p can neither have a maximum nor a minimum inside the range of h. Thus, if we define an h-band to be the closure of a connected component of the preimage under p of the open interval (h_{min}, h_{max}),[3] then $\sigma(H)$ is the union of N h-bands which may pairwise intersect at most in one point. At such an intersection point, we must have $p(E) \in \{h_{min}, h_{max}\}$

[3] reserving the name band for the connected components of the spectrum which contain at least an interval

and $p'(E) = 0$. If that happens, we call it h-band touching. An h-band edge is a value of E at which $p(E) \in \{h_{min}, h_{max}\}$.

If no h-band touching occurs then the family of eigenvalue equations $H_{\vec\theta}\psi_{\vec\theta} = E(\vec\theta)\psi_{\vec\theta}$, $\psi_{\vec\theta} \in \mathcal{H}_{\vec\theta}$ defines N continuous functions $E : S^1 \times S^1 \to \mathbb{R}$. These are called band functions. Their image coincides with the spectrum. Away from Bloch-parameters $\vec\theta \in S^1 \times S^1$ where h-band touching occurs these functions may in fact always be defined. But if one tries to extend them over these points topological effects may appear. If the set of Bloch-parameters where h-band touching occurs is discrete there will be conical singularities in the manifold which is defined by the graph of the E's. This is for instance the case for the Hofstadter Hamiltonian at $E = 0$ with rational flux having even denominator.

3 Models of Hofstadter Type and Quantum Integrals

Above we have explained the significance of the Weyl-operators for the Landau model and the rôle of the Hofstadter Hamiltonian for its discretization. We now first introduce the family of operators to be investigated in this article. They are generalizations of the Hofstadter Hamiltonian involving translations to the next nearest neighbour. We then establish that these operators are invariant under certain automorphisms of the Weyl-Heisenberg algebra. Finally we show that all operators considered satisfy a Chambers relation and compute their off-set functions.

3.1 Hamiltonians of Hofstadter type

The (isotropic) Hofstadter Hamiltonian is, up to a constant, the discrete magnetic Laplacian. The representation of $\mathcal{A}_\gamma$ by discrete magnetic translations being faithful we consider that Hamiltonian abstractly as an element of $\mathcal{A}_\gamma$, the Hofstadter Hamiltonian then becoming $H = W(1,0) + W(0,1) + W(-1,0) + W(0,-1)$. The family of selfadjoint elements which we consider in this article is given by

$$H(a,b,c,d) = a\,W(1,0) + b\,W(0,1) + c\,W(1,1) + d\,W(1,-1) + h.c.$$

depending on complex parameters a, b, c, d which may depend on the flux. The expression $X + h.c.$ in a star algebra shall mean $X + X^*$. For simplicity we restrict our attention to the case $c \neq 0, d \neq 0$ which seems to exclude the Hofstadter Hamiltonian. But in fact, $H(0,0,1,d)$, $d > 0$, which belongs to $\mathcal{A}_{2\gamma} \subset \mathcal{A}_\gamma$ is algebraically equivalent to the anisotropic Hofstadter Hamiltonian with doubled flux. If we rescale $H(a,b,c,d)$ by a real nonzero factor then its spectrum will be rescaled by the same factor. If we multiply u and v by complex numbers z_1, z_2 of modulus 1 then the spectrum doesn't change at all. Restricting to the case $c \neq 0, d \neq 0$ we therefore may take the freedom to fix the following "normalization": We set

$$c = k$$

where k is a strictly positive real number, and

$$d = c^{-1},$$

denoting the Hamiltonian then more briefly by $H(a,b,k)$. For clarity we repeat its expression in terms of u and v

$$H(a,b,k) = au + bv + kq^{\frac{1}{2}}uv + k^{-1}q^{-\frac{1}{2}}uv^{-1} + h.c. \tag{3.1}$$

3.2 Automorphisms of $\mathcal{A}_\gamma$ and quantum integrals

Given an automorphism α of $\mathcal{A}_\gamma$ we call a selfadjoint operator H which is invariant under α, i.e. for which $\alpha(H) = H$, simply a quantum integral (for α). This notion is based on the interpretation of α as time 1 evolution of a discrete dynamical system. H is then an integral of motion and we abreviate this as quantum integral since the dynamics is not defined on a space or a commutative C^*-algebra but rather on a non commutative C^*-algebra.

Examples of automorphisms of $\mathcal{A}_\gamma$ are those which are given by conjugation with unitary elements $U \in \mathcal{A}_\gamma$, $x \mapsto UxU^*$. They are called inner. Of another class of automorphisms we have already made use above: given two complex numbers z_1, z_2 of modulus 1, $u \mapsto z_1 u$, $v \mapsto z_2 v$ extends to an automorphisms of $\mathcal{A}_\gamma$. Furthermore, there is a group homomorphism between $SL(2,\mathbb{Z})$ and the group of automorphisms of $\mathcal{A}_\gamma$ given by

$$M \mapsto \alpha_M, \quad \alpha_M(W(n)) = W(Mn). \tag{3.2}$$

The case $\mathcal{F} = \alpha_M$ with

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is of particular importance for us. Raised to the fourth power it is the identity. For rational $\frac{\gamma}{2\pi}$, $\mathcal{F}$ has the interpretation of the discrete Fouriertransform [**?**]. Another family, labelled by continuous functions $f : S^1 \to S^1$, is of importance for us. Recall that, by spectral calculus, a continuous function $f : S^1 \to S^1$ defines a function from the group of unitaries of $\mathcal{A}_\gamma$ into itself.

Lemma 1 *Let $f : S^1 \to S^1$ be a continuous function. Then*

$$\begin{aligned} \alpha_f : \mathcal{A}_\gamma &\to \mathcal{A}_\gamma \\ u &\mapsto u \\ v &\mapsto vf(u) \end{aligned} \tag{3.3}$$

is an automorphism.

Proof: Having defined α_f on the generators of $\mathcal{A}_\gamma$ we have to check, first, whether α_f preserves the relation $uv - q^{-1}vu = 0$, and second, whether it is invertible. As for the first, $\alpha_f(uv - q^{-1}vu) = (uv - q^{-1}vu)f(u) = 0$. For the second, one immediately checks that $u \mapsto u$, $v \mapsto vf(u)^*$ is the inverse of α_f. q.e.d.

Theorem 3 *$H(a,a,k)$ is a quantum integral for $\breve{\alpha}_{f_a} := \mathcal{F} \circ \alpha_{f_a}$ with*

$$f_a(z) = \frac{\bar{a} + kq^{\frac{1}{2}} z^{-1} + k^{-1} q^{-\frac{1}{2}} z}{a + k^{-1} q^{\frac{1}{2}} z^{-1} + kq^{-\frac{1}{2}} z}. \tag{3.4}$$

$H(a,b,k)$ is a quantum integral for $\breve{\alpha}_{f_a} \circ \breve{\alpha}_{f_b}$.

Proof: Let us compute $\breve{\alpha}_f(H(a,a,k))$ for arbitrary continuous $f : S^1 \to S^1$. Then

$$\begin{aligned}\breve{\alpha}_f(H(a,a,k)) &= av + au^{-1} f(v) + kq^{\frac{1}{2}} vu^{-1} f(v) + k^{-1} q^{-\frac{1}{2}} v f^*(v) u + h.c. \\ &= av + \left(\bar{a} + kq^{\frac{1}{2}} v^* + k^{-1} q^{-\frac{1}{2}} v\right) f^*(v) u + h.c.\end{aligned}$$

Comparing with (**??**) we see that $\breve{\alpha}_f(H(a,a,k)) = H(a,a,k)$ if

$$\left(\bar{a} + kq^{\frac{1}{2}} v^* + k^{-1} q^{-\frac{1}{2}} v\right) f^*(v) = a + kq^{-\frac{1}{2}} v + +k^{-1} q^{\frac{1}{2}} v^{-1};$$

hence if f is as in (**??**). Note that f_a is the quotient of two complex conjugate numbers. If the denominator has a zero the nominator has one too and we may obtain the value of f at that zero by continuity. This proves the first statement. The second statement follows from

$$\begin{aligned}\breve{\alpha}_{f_b}(H(a,b,k)) &= \breve{\alpha}_{f_b}(H(b,b,k)) + (a-b)\breve{\alpha}_{f_b}(u + u*) \\ &= H(b,b,k) + (a-b)(v + v*) \\ &= H(b,a,k).\end{aligned}$$

q.e.d.

Note that the automorphism as constructed in Sect. 5 will be the inverse of the above automorphism, i.e. $\breve{\alpha}_{f_a}^{-1} = \alpha_{f_a^{-1}} \circ \mathcal{F}^3$. Moreover in Sect. **??** we will consider the case $a, b \in \mathbb{R}$ which implies that $\mathcal{F}^2(H(a,b,k)) = H(a,b,k)$, hence there $H(a,a,k)$ is also a quantum integral for $\alpha_{f_a^{-1}} \circ \mathcal{F}$.

It is not clear how one can make use of the invariance of $H(a,b,k)$ under such automorphisms in general. However, in case the dynamical system provided by the automorphism is integrable (in a sense specified in Sect. **??**) one can use the algebraic Bethe ansatz to calculate at least some eigenvalues and -functions of $H(a,b,k)$.

Remark: It is not clear to us what the rôle of the above automorphism in solid state physics is. It is certainly not the case that $H(a,b,k)$ generates the automorphism in the sense that $\breve{\alpha}_{f_a} \circ \breve{\alpha}_{f_b}(x) = U(H)xU(H)^*$ where $U(H)$ is a unitary which is a continuous function of H, as e.g. $U(H) = \exp itH(a,b,k)$ for some $t \in \mathbb{R}$. In fact, this would mean that $\breve{\alpha}_{f_a} \circ \breve{\alpha}_{f_b}$ is inner which in the case of irrational flux is ruled out by a topological obstruction. We cannot explain this obstruction here, but see [**?**].

3.3 Chambers relation for $H(a,b,k)$

Computing the off-set function for $H(a,b,k)$ we follow so closely [?], where the case $a = b \in \mathbb{R}$ is treated, that we do not repeat the explicit computation here. Since $H(a,b,k)$ contains each generator of $\mathcal{A}_\gamma$ at most with absolute power 1 and since the characteristic polynomial $\det(H_{\vec\theta}(a,b,k)-\lambda)$ is invariant under the transformation $\theta_i \mapsto \theta_i + n_i\gamma$, $n_i \in \mathbb{Z}$, one finds that, for $\frac{\gamma}{2\pi} = \frac{M}{N}$,

$$\det(H_{\vec\theta}(a,b,k)-\lambda) = p(a,b,k;\lambda) + h(a,b,k;\vec\theta)$$

where $p(a,b,k;0) = 0$ and, setting $\xi_j = e^{i\theta_j}$,

$$\begin{aligned} h(a,b,k;\vec\theta) &= h_1(a,b,k)\xi_1^N + h_2(a,b,k)\xi_2^N \\ &\quad + h_3(a,b,k)\xi_1^N\xi_2^N + h_4(a,b,k)\xi_1^N\xi_2^{-N} + c.c. \end{aligned}$$

(c.c. standing for complex conjugate). The coefficients are determined by considering various limits $\xi_1 \to \infty$ or $\xi_2 \to \infty$ in that equation. The result is

$$\begin{aligned} h_1(a,b,k) &= -2T_N(-\frac{a}{2}) \\ h_2(a,b,k) &= -2T_N(-\frac{b}{2}) \\ h_3(a,b,k) &= (-1)^M k^N \\ h_4(a,b,k) &= (-1)^M k^{-N}, \end{aligned}$$

where T_N is up to constants the Nth Chebychev polynomial of the second kind,

$$T_N(\frac{z+z^{-1}}{2}) = \frac{z^N + z^{-N}}{2},$$

for complex $z \neq 0$. Recall that the classical symbol of the Weyl-operator $W(f)$ associated to a test function $f : \mathbb{Z}^2 \to \mathbb{C}$ is the Fouriertransform of f. Hence the classical symbol of $H(a,b,k)$ is $h_{class.}(a,b,k;z_1,z_2) = az_1 + bz_2 + kq^{\frac{1}{2}}z_1z_2 + k^{-1}q^{-\frac{1}{2}}z_1z_2^{-1} + c.c.$. Thus we see that the off-set function $h(a,b,k;\theta)$ of $H(a,b,k)$ is obtained by inserting $z_j = e^{iN\theta_j}$ in the classical symbol after having replaced the coefficients a, b, and k by $-2T_N(-\frac{a}{2})$, $-2T_N(-\frac{b}{2})$, and $(-1)^M k^N$, respectively.

We end this section by formulating the off-set function and its gradient in variables r_a, θ_a and r_b, θ_b where

$$a = r_a e^{i\theta_a} + r_a^{-1} e^{-i\theta_a}$$

with $r_a \geq 1$ and alike for b. Apart from the case $a = 0$, in which $r_a = 1$ and $\theta_a = 0$ or π, this determines r_a and θ_a uniquely.

$$\begin{aligned} h(a,b,k;\theta) &= 2(-1)^{N-1}\left(r_a^N \cos N(\theta_1+\theta_a) + r_a^{-N}\cos N(\theta_1-\theta_a)\right) \\ &\quad + 2(-1)^{N-1}\left(r_b^N \cos N(\theta_2+\theta_b) + r_b^{-N}\cos N(\theta_2-\theta_b)\right) \\ &\quad + 2(-1)^M\left(k^N \cos N(\theta_1+\theta_2) + k^{-N}\cos N(\theta_1-\theta_2)\right), \end{aligned} \quad (3.5)$$

$$\partial_{\theta_1} h(a,b,k;\vec{\theta}) = 2(-1)^N N\left(r_a^N \sin N(\theta_1+\theta_a) + r_a^{-N}\sin N(\theta_1-\theta_a)\right) \qquad (3.6)$$
$$+\, 2(-1)^{M-1}N\left(k^N \sin N(\theta_1+\theta_2) + k^{-N}\sin N(\theta_1-\theta_2)\right),$$
$$\partial_{\theta_2} h(a,b,k;\vec{\theta}) = 2(-1)^N N\left(r_b^N \sin N(\theta_2+\theta_b) + r_a^{-N}\sin N(\theta_2-\theta_b)\right) \qquad (3.7)$$
$$+\, 2(-1)^{M-1}N\left(k^N \sin N(\theta_1+\theta_2) - k^{-N}\sin N(\theta_1-\theta_2)\right).$$

4 Polynomial Solutions to the Eigenvalue Equation

Wiegmann and Zabrodin proposed to study the spectrum of the Hofstadter Hamiltonian by a method they called functional (or polynomial) Bethe ansatz [?] refering to an article of Sklyanin [?]. Their approach is based upon the observation that, for rational flux, the Hofstadter Hamiltonian may be expressed through the generators of the quantum group $U_q(sl_2)$, $q = e^{\frac{i\gamma}{2}}$, represented irreducibly on a subspace of $L^2(S^1)$. Strictly speaking, it is a representation of the Weyl-Heisenberg algebra on $L^2(S^1)$ in which the Hofstadter Hamiltonian is thus expressed. Therefore, the above identification of the Hamiltonian with an element of $U_q(sl_2)$ in an irreducible representation of the quantum group has the same effect than restricting the Hamiltonian to an irreducible component of the representation of the Weyl-Heisenberg algebra. The Bethe ansatz equations then furnish, if solved, eigenfunctions to eigenvalues corresponding to points in the inner part of the bands. It is not clear to us what precisely the rôle of the quantum group in this approach is, since we obtain below the Bethe ansatz equations and energies by a simple ansatz for polynomial eigenfunctions in $L^2(S^1)$ without making use of the quantum group structure. The question after the existence of polynomial eigenfunctions for difference operators was also addressed in [?], the authors making use of $U_q(sl_2)$ aiming at a classification of difference equations which preserve a space of polynomial functions.

Let $\mathcal{P}_{m_1,m_2}$, $m_1 \le m_2 \in \mathbb{Z}$, be the subspace of $L^2(S^1)$ which is generated by the monomials $\{z^m | m_1 \le m \le m_2\}$. A Laurent-polynomial function of $L^2(S^1)$ is a function which is contained in some $\mathcal{P}_{m_1,m_2}$. Note that in the Fourierspace $\ell^2(\mathbb{Z})$ of $L^2(S^1)$ these correspond to functions with finite support. The polynomial Bethe ansatz is an ansatz for Laurent-polynomial eigenfunctions of an operator H in a representation on $L^2(S^1)$. The representations on $L^2(S^1)$ considered here are Weyl-Schrödinger representations, cf. (**??**). The next theorem establishes necessary and sufficient criterions for the existence of polynomial solutions for our family of operators and in the forthcoming subsections we discuss, for rational flux, where in the bands their corresponding eigenvalues lie.

In the Weyl-Schrödinger representation (**??**) with angle θ, $H(a,b,k)$ acts on a function $f \in L^2(S^1)$ as

$$H(a,b,k)\cdot f(z) = e^{i\theta}(a + kq^{-\frac{1}{2}}z + k^{-1}q^{\frac{1}{2}}z^{-1})f(q^{-1}z) \qquad (4.1)$$
$$+\,(bz + \bar{b}z^{-1})f(z) + e^{-i\theta}(\bar{a} + kq^{-\frac{1}{2}}z^{-1} + k^{-1}q^{\frac{1}{2}}z)f(qz).$$

Theorem 4 *The operator $H(a,b,k)$ acting in representation (**??**) on $L^2(S^1)$ with angle θ preserves $\mathcal{P}_n := \mathcal{P}_{0,n-1}$, $n \in \mathbb{N}$, if and only if*

$$(q^n - 1)(k^2 e^{2i\theta} - q^{n-1}) = 0 \tag{4.2}$$

and

$$b = -k e^{i\theta} q^{\frac{1}{2}} - k^{-1} e^{-i\theta} q^{-\frac{1}{2}}. \tag{4.3}$$

(The latter equation is equivalent to $r_b = k$ and $\theta = \theta_b - \frac{\gamma}{2} + \pi$.) If $H(a,b,k)$ doesn't preserve $\mathcal{P}_{n'}$ with $n' < n$ then all its eigenfunctions in $\mathcal{P}_n$ are given by

$$f(z) = \prod_{j=1}^{n-1} (z - z_j),$$

where the z_j satisfy the Bethe ansatz-equations

$$e^{2i\theta} \frac{a + kq^{-\frac{1}{2}} z_j + k^{-1} q^{\frac{1}{2}} z_j^{-1}}{\bar{a} + kq^{-\frac{1}{2}} z_j^{-1} + k^{-1} q^{\frac{1}{2}} z_j} = -\prod_{i=1}^{n-1} \frac{q z_j - z_i}{q^{-1} z_j - z_i}. \tag{4.4}$$

The corresponding eigenvalues are given by

$$\begin{aligned} E(z_1, \ldots, z_{n-1}) &= e^{i\theta} q^{1-n} a + e^{-i\theta} q^{n-1} \bar{a} \\ &\quad - (b + e^{i\theta} k q^{\frac{3}{2}-n} + e^{-i\theta} k^{-1} q^{n-\frac{3}{2}}) \sum_{j=1}^{n-1} z_j. \end{aligned} \tag{4.5}$$

Proof: Inserting the function $f(z) = z^m$ in (**??**) one obtains

$$\begin{aligned} H(a,b,k) \cdot f(z) &= (a e^{i\theta} q^{-m} + \bar{a} e^{-i\theta} q^m) z^m \\ &\quad + (b + k e^{i\theta} q^{-m-\frac{1}{2}} + k^{-1} e^{-i\theta} q^{m+\frac{1}{2}}) z^{m+1} \\ &\quad + (\bar{b} + k e^{-i\theta} q^{m-\frac{1}{2}} + k^{-1} e^{i\theta} q^{-m+\frac{1}{2}}) z^{m-1}. \end{aligned}$$

Thus the condition for the existence of polynomial eigenfunctions is equivalent to the condition that $H(a,b,k)$ preserves a subspace $\mathcal{P}_{m_1,m_2}$, namely it is

$$b + k e^{i\theta} q^{-m_2-\frac{1}{2}} + k^{-1} e^{-i\theta} q^{m_2+\frac{1}{2}} = 0 \tag{4.6}$$

$$\bar{b} + k e^{-i\theta} q^{m_1-\frac{1}{2}} + k^{-1} e^{i\theta} q^{-m_1+\frac{1}{2}} = 0. \tag{4.7}$$

Setting $m_1 = 0$ and $n = m_2 + 1$ (**??**,**??**) are equivalent to (**??**,**??**). Thus if (**??**,**??**) hold true then $H(a,b,k)$ preserves $\mathcal{P}_n$ and we can make the ansatz $f(z) = \prod_{j=1}^{n-1}(z - z_j)$ and substitute it in the eigenvalue equation (**??**). In fact, if $H(a,b,k)$ doesn't preserve $\mathcal{P}_{n'}$ with $n' < n$ then all its eigenfunctions of $\mathcal{P}_n$ have to be polynomials of degree $n-1$. Dividing the resulting equation by $f(z)$

one obtains

$$E = (bz + \bar{b}z^{-1}) + e^{i\theta}(a + kq^{-\frac{1}{2}}z + k^{-1}q^{\frac{1}{2}}z^{-1}) \prod_{j=1}^{n-1} \frac{q^{-1}z - z_j}{z - z_j} \tag{4.8}$$

$$+ e^{-i\theta}(\bar{a} + kq^{-\frac{1}{2}}z^{-1} + k^{-1}q^{\frac{1}{2}}z) \prod_{j=1}^{n-1} \frac{qz - z_j}{z - z_j}.$$

This equation can only hold true, if the r.h.s. is constant in z. This implies that the poles which seem to be apparent at $z = 0, \infty, z_1, \ldots, z_{n-1}$ have to cancel out. Cancellation of the poles at $z = 0$ and $z = \infty$ follows from (**??**,**??**). The pole at $z = z_j$ cancels out only if, for all $1 \leq j \leq n-1$

$$e^{2i\theta} \frac{a + kq^{-\frac{1}{2}}z_j + k^{-1}q^{\frac{1}{2}}z_j^{-1}}{\bar{a} + kq^{-\frac{1}{2}}z_j^{-1} + k^{-1}q^{\frac{1}{2}}z_j} = -\prod_{i=1}^{n-1} \frac{qz_j - z_i}{q^{-1}z_j - z_i}. \tag{4.9}$$

These are $n-1$ equations determining the possible values of z_j and thus the eigenfunction. Their corresponding eigenvalues are obtained by comparison of the zero order term, or, what amounts to the same, upon remultiplying equation (**??**) by $\prod_{j=1}^{n-1}(z - z_j)$ and comparing the term of order z^{n-1}. This yields (**??**). q.e.d.

We call the eigenvalue (**??**) and the corresponding eigenvector of $H(a, b, k)$ simply Bethe-ansatz eigenvalue and Bethe-ansatz eigenvector, respectively. We have established a neccessary and sufficient condition for H to have eigenfunctions in $\mathcal{P}_{m_1,m_2}$ for $m_1 = 0$. The extension of this result to arbitrary m_1 is a simple modification. In fact, if H acting in the Weyl-Schrödinger representation with angle θ preserves $\mathcal{P}_{m_2-m_1+1}$ then it preserves $\mathcal{P}_{m_1,m_2}$ in the Weyl-Schrödinger representation with angle $\theta + m_1\gamma$. Since changing θ into $\theta + m\gamma$, $m \in \mathbb{Z}$, changes the representation into a unitary equivalent one, essential properties like the Bethe ansatz-eigenvalues remain unchanged. Moreover, if H does preserve as well $\mathcal{P}_{m_1,m_2'}$ with $m_1 < m_2' < m_2$ then the eigenvectors obtained form the Bethe ansatz-equations are of the form $f(z) = z^{m_2'} \prod_{j=1}^{m_2-m_2'}$. In particular they all lie in $\mathcal{P}_{m_2',m_2}$.

There are two possibilities to meet the first part (**??**) of the requirements of the theorem leading to two qualitatively different situations. Either one specializes the flux to values for which the first factor of (**??**) vanishes or, in case $k = 1$, relates the angle θ to the flux in such a way that the second factor vanishes. The two corollaries below treat these two cases. The second case distinguishes from the first in two remarkable ways. First, the eigenvalues obtained (may be chosen to) depend analytically on the flux – so that in particular they hold as well for irrational flux –, and second, the behaviour of the off-set function at the Bloch-parameters for which Bethe ansatz-eigenvalues exist is different. In the most interesting case of the Quantum pendulum, which is discussed in Section **??**, the off-set function is at these special values extremal. Hence, the set of Bloch-parameters in question is critical. Even more, this set is a circle.

4.0.1 *Positioning of Bethe ansatz-eigenvalues in bands*

For rational flux our operators have a band spectrum. Having established polynomial solutions for certain parameters of our family of operators it is therefore natural to ask, for rational flux, where in the bands they lie. To solve this question one has to, first, determine Bloch-parameters for which the Bethe ansatz-eigenvalue exists, second, evaluate the off-set function at these Bloch-parameters and compare its value with h_{min} and h_{max}. But, third, one would have to take the preimage of these three values under p_γ to obtain the position of the Bethe ansatz-eigenvalue in the h-band. We shall carry out the first two steps, however, we do not have good enough controle over p_γ to carry out the third step. Without that last step, one does not know precisely where and in which band the Bethe ansatz-eigenvalue lies but one still has a topological picture of its relative position inside the band it belongs to.

A qualitative characterisation of the set of Bloch-parameters at which polynomial eigenfunctions may exist is given by the next lemma.

Lemma 2 *Let $H \in \mathcal{A}_\gamma$ be an element which satisfies a Chambers relation with off set function h. Then, for rational $\frac{\gamma}{2\pi}$, a neccessary condition for H to have an eigenfunction in representation (**??**) on $L^2(S^1)$ with angle θ is that $h(\theta,\theta_2)$ is independent of θ_2. In particular, if the eigenvalue lies at an h-band edge then $h(\theta,\theta_2)$ belongs to an absolute maximum or minimum which is taken on a set containing a whole circle.*

Proof: Let $\frac{\gamma}{2\pi} = \frac{M}{N}$ and assume that $f \in L^2(S^1)$ satisfies $H \cdot f = Ef$ for some E, H acting in the Weyl-Schrödinger representation with angle θ. Since $\frac{\gamma}{2\pi}$ is rational, this representation is reducible and we can decompose it as described in Section 1. For that let $\phi \in \ell^2(\mathbb{Z})$ be the Fouriertransform of f. Then $H \cdot \phi = E\phi$, H acting in representation (**??**) with angle θ. Recall from Section 1 that the component $\hat{\phi}_\varphi$ of $\mathcal{F}_2(\phi)$ in the fibre $\mathcal{H}_{(\theta,\varphi)}$ satisfies (**??**) and hence $\hat{\phi}_\varphi(l) = \sum_{n\in\mathbb{Z}} e^{in\varphi} v^{-nN} \cdot \phi(l)$. Since v^N lies in the centre of A_γ,

$$H \sum_{n\in\mathbb{Z}} e^{in\varphi} v^{-nN} \cdot \phi(l) = E \sum_{n\in\mathbb{Z}} e^{in\varphi} v^{-nN} \cdot \phi(l) = E\hat{\phi}_\varphi(l)$$

independently of φ. Therefore, $\hat{\phi}_\varphi$ is an eigenvector to eigenvalue E of the operator which represents H in the irreducible representation space $\mathcal{H}_{(\theta,\varphi)}$. Consequently, E is a zero of $\det(H_{\theta,\theta_2} - \lambda)$, $N\theta_2 = \varphi$, for all φ and hence $h(\theta,\theta_2)$ must be constant in θ_2. The last statement is clear. q.e.d.

The lemma shows, first, that polynomial solutions in Weyl-Schrödinger representations cannot occur everywhere in bands but are related to the form of the off-set function. It also shows that we do not have to decompose the Weyl-

Schrödinger representation into irreducible ones in order to obtain the Bloch-parameters at which Bethe ansatz-eigenvalues exist. In fact, one of the Bloch-parameters is always the θ which enters in Theorem **??** and since $h(\theta, \theta_2)$ is independent of θ_2 the other Bloch-parameter is not constraint, i.e. the set of Bloch-parameters at which a Bethe ansatz-eigenvalue exists is $\{(\theta, \theta_2) | \theta_2 \in S^1\}$.

4.1 Polynomial solutions for rational flux

Corollary 1 *If $\frac{\gamma}{2\pi} = \frac{M}{N}$ and $k = r_b$, then $H(a, b, k)$ acting in representation (**??**) on $L^2(S^1)$ with angle $\theta = \pi + \theta_b - \frac{\gamma}{2}$ preserves $\mathcal{P}_N$. The eigenvalues obtained from a solution $\{z_j\}_{j=1,\ldots,N-1}$ of the Bethe ansatz-equations, now becoming*

$$e^{2i\theta_b} \frac{a + kq^{-\frac{1}{2}} z_j + k^{-1} q^{\frac{1}{2}} z_j^{-1}}{\bar{a} + kq^{-\frac{1}{2}} z_j^{-1} + k^{-1} q^{\frac{1}{2}} z_j} = - \prod_{i=1}^{N-1} \frac{q z_j - z_i}{z_j - q z_i}, \tag{4.10}$$

are then given by

$$\begin{aligned} E(z_1, \ldots, z_{N-1}) = &-e^{i(\theta_b + \frac{\gamma}{2})} a - e^{-i(\theta_b + \frac{\gamma}{2})} \bar{a} \\ &+ 2i(e^{i(\theta_b + \frac{\gamma}{2})} r_b - e^{-i(\theta_b + \frac{\gamma}{2})} r_b^{-1}) \sin \frac{\gamma}{2} \sum_{j=1}^{N-1} z_j. \end{aligned} \tag{4.11}$$

These values of the energy belong to the inner part of the bands.

Proof: Apart from the statements about the positioning of the Bethe ansatz eigenvalues the corrollary follows directly form the above theorem if one takes its hypothesis into account, in particular that $q^N = 1$. To show that the Bethe ansatz-eigenvalues are never h-band edges suppose that they were. Then $h(a, b, r_b; \pi + \theta_b - \frac{\gamma}{2}, \theta_2)$ would be an extremum for all θ_2, according to Lemma **??**. But by (**??**), we see that $\partial_{\theta_1} h$ evaluated at $(\pi + \theta_b - \frac{\gamma}{2}, \theta_2)$ does not vanish for all values of θ_2. This is a contradiction. q.e.d.

For later use we state

$$h(a, b, r_b; \pi + \theta_b - \frac{\gamma}{2}, \theta_2) = 2(-1)^{M-1} \left(r_a^N \cos N(\theta_b + \theta_a) + r_a^{-N} \cos N(\theta_b - \theta_a) \right) \tag{4.12}$$

which, as it should be, is independent of θ_2.

The formula for the eigenvalues requires a solution of the Bethe ansatz-equations. This is a rather difficult task. On the other hand, it is a direct consequence of the corollary that these eigenvalues are among those[4] of the $N \times N$-matrix $\check{H}(a, b)$ having coefficients

$$\check{H}_{lm}(a, b) = (z^l, H(a, b, r_b) \cdot z^m), \quad 0 \le l, m \le N - 1, \tag{4.13}$$

[4] All eigenvalues of $\check{H}(a, b)$ are given by (**??**) in case $H(a, b, 1)$ does not preserve a $\mathcal{P}_{n'}$ with $n' < n$.

$H(a,b,r_b)$ acting in the representation with angle $\theta = \pi + \theta_b - \frac{\gamma}{2}$. In fact, this matrix coincides with the matrix $H_{\vec{\theta}}(a,b,r_b)$ obtained in the irreducible representation (**??**,**??**) of $\mathcal{A}_\gamma$ with Blochparameters $\theta_1 = \pi + \theta_b - \frac{\gamma}{2}$ and arbitrary θ_2. Its diagonalization its much easier than solving the Bethe ansatz-equations. Hence, from the computational point of view the functional Bethe ansatz is a step backwards. But the hope is that the Bethe ansatz-equations simplify enormously if one considers the limit $N \to \infty$. This could then be used for investigating the eigenvalue equation for irrational values of the flux, the irrational being approximated by rationals with increasing denominators.

4.2 Polynomial solutions depending analytically on the flux

Corollary 2 *If $k = 1$ then the operator $H(a,b,k)$ acting in representation (**??**) on $L^2(S^1)$ with angle $\theta = \pi + \frac{n-1}{2}\gamma$ preserves $\mathcal{P}_n$, $n \in \mathbb{N}$, provided*

$$b = 2\cos\frac{n\gamma}{2}. \tag{4.14}$$

The eigenvalues obtained from a solution $\{z_j\}_{j=1,\ldots,N-1}$ of the Bethe ansatz-equations, now becoming

$$\frac{a + kq^{-\frac{1}{2}}z_j + k^{-1}q^{\frac{1}{2}}z_j^{-1}}{\bar{a} + kq^{-\frac{1}{2}}z_j^{-1} + k^{-1}q^{\frac{1}{2}}z_j} = -\prod_{i=1}^{n-1}\frac{qz_j - z_i}{z_j - qz_i}, \tag{4.15}$$

are then given by

$$E(z_1,\ldots,z_{n-1}) = -q^{\frac{1-n}{2}}a - q^{\frac{n-1}{2}}\bar{a} - 4\sin\frac{\gamma}{2}\sin\frac{n-1}{2}\gamma\sum_{j=1}^{n-1}z_j. \tag{4.16}$$

All eigenvalues and eigenvectors of the operator which lie in $\mathcal{P}_n$ may be chosen to depend analytically on $q^{\frac{1}{2}}$ for $q^{\frac{1}{2}}$ belonging to the punctured S^1. Furthermore, for rational $\frac{\gamma}{2\pi}$, not only the off-set function itself but also its gradient is constant along the circle of Bloch-parameters at which these Bethe ansatz-eigenvalues exist. If moreover a is real, this circle is a critical set of the off-set function.

Proof: The first statements follow directly from the above theorem if one specifies the values for b, k and θ as stated. The eigenvalues of $H(a, 2\cos\frac{n\gamma}{2}, 1)$ obtained by the Bethe ansatz-ansatz as above, i.e. by (**??**), are among the eigenvalues of the $n \times n$ matrix

$$\tilde{H}_{lm}(a,n) = (z^l, H(a, 2\cos\frac{n\gamma}{2}, 1)\cdot z^m), \quad 0 \le l, m \le n-1, \tag{4.17}$$

$H(a, 2\cos\frac{n\gamma}{2}, 1)$ acting in the representation with angle $\theta = \pi + \frac{n-1}{2}\gamma$, and the eigenvectors determined by (**??**) are related to the eigenvectors for that matrix in the obvious way. But the matrix $\tilde{H}(a,n)$ depends analytically on $q^{\frac{1}{2}}$. The existence of eigenvectors and eigenvalues depending analytically on $q^{\frac{1}{2}}$ on a simply connected domain follows therefore from the analysis of [?]. To proof the remaining statement we compute the derivative of the off-set function at the circle $\{(\pi + \frac{n-1}{2}\gamma, \theta_2) | \theta_2 \in S^1\}$. With $r_b = 1$ and $\theta_b = \frac{n}{2}\gamma$ we obtain from (**??**)

$$\partial_{\theta_1} h(a, 2\cos\frac{n}{2}\gamma, 1; \pi + \frac{n-1}{2}\gamma, \theta_2) = 2N(-1)^{M(n-1)}\left(r_a^N - r_a^{-N}\right)\sin N\theta_a. \tag{4.18}$$

Since this is independent of θ_2 and since the other partial derivative vanishes by Lemma **??**, the gradient is constant along the above circle. The gradient vanishes if $r_a = 1$ or $\theta_a = 0, \pi$, or equivalently, if a is real. q.e.d.

Also here let us state the value of the off-set function

$$h(a, 2\cos\frac{n}{2}\gamma, 1; \pi + \frac{n-1}{2}\gamma, \theta_2) = 2(-1)^{M(n-1)-1}\left(r_a^N + r_a^{-N}\right)\cos N\theta_a. \tag{4.19}$$

As in the previous case, the Bethe ansatz-equations are more difficult to solve than the eigenvalue equation of $\tilde{H}(a,n)$.

Note that $\theta = \frac{n-1}{2}\gamma$ is for $k = 1$ also a solution of (**??**), but we may always absorbe the phase -1 simultanuously in u and v without changing the spectrum so that we may restrict our attention to the above case.

4.3 Application to the Hofstadter Hamiltonian and QP-integrals

4.3.1 *Hofstadter Hamiltonian*

As already mentioned, for $a = b = 0$, $H(a,b,k)$ corresponds to the anisotropic Hofstadter model with flux 2γ. To make this more clear recall that the anisotropic Hofstadter Hamiltonian with flux $\check{\gamma}$ is given by $H_{Hof}(k,\check{\gamma}) := \check{u} + k^{-1}\check{v} + h.c.$, $\check{u}$ and $\check{v}$ generating $\mathcal{A}_{\check{\gamma}}$, i.e. $\check{u}\check{v} = e^{-i\check{\gamma}}\check{v}\check{u}$. Then, upon writing $\check{u} = e^{-i\frac{\gamma}{2}}uv$ and $\check{v} = e^{-i\frac{\gamma}{2}}uv^*$ where u, v generate $\mathcal{A}_\gamma$, $\gamma = \frac{\check{\gamma}}{2}$, we have

$$H_{Hof}(k^2, 2\gamma) = k^{-1}H(0,0,k).$$

For $a = b = 0$ we can take $r_a = r_b = 1$ and $\theta_a = \theta_b = \pm\frac{\pi}{2}$. In particular, Corollary **??** can be applied to the isotropic Hofstadter Hamiltonian, for which $k = 1$. Corollary **??** is not applicable since it requires $b \neq 0$. As a result, $H(0,0,1)$ preserves $\mathcal{P}_N$, for $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M,N) = 1$, but no $\mathcal{P}_{N'}$ with $N' < N$. Hence, according to Corollary **??** it has eigenfunctions of the form $\prod_{j=1}^{N-1}(z - z_j)$ where the z_j satisfy

$$\frac{q^{-\frac{1}{2}}z_j + q^{\frac{1}{2}}z_j^{-1}}{q^{-\frac{1}{2}}z_j^{-1} + q^{\frac{1}{2}}z_j} = \prod_{i=1}^{N-1}\frac{qz_j - z_i}{z_j - qz_i}. \tag{4.20}$$

The corresponding eigenvalues are given by

$$E(z_1, \ldots, z_{n-1}) = \pm 2\sin\gamma\sum_{j=1}^{n-1} z_j. \tag{4.21}$$

(The sign depends on the choice for θ_b.) If N is odd then (**??**) specializes to $h(0,0,1,\frac{\pi-\gamma}{2},\theta_2) = 0$. Since, for the case at hand, $h_{min} = -h_{max}$ this is sometimes interpreted as being the middle of the band. We call the corresponding eigenvalues therefore also "middle points", although, strictly speaking, they lie in the middle of the band only in the case of odd N and energy value $E = 0$.

For odd N equations (**??**,**??**) where obtained in [**?**]. It is worth mentioning that Bethe ansatz-eigenfunctions can be obtained for $E = 0$ and arbitrary odd N [**?**]. Zabrodin has generalized the approach of [**?**] to the anisotropic Hofstadter Hamiltonian by considering an auxiliary Hamiltonian which turns out to have the same spectrum in the appropriate representations [**?**]. Faddeev and Kashaev present a rather involved different approach to the problem of solving the spectra of models of Hofstader-type [**?**]. They obtain a class of Bethe ansatz-equations which includes those of [**?**] as well as Bethe ansatz-equations for the isotropic Hofstadter with even N.

One might think that it could be more successful to consider directly the operator $H(1,1,0,0)$ which is also the isotropic Hofstadter but with flux γ and which we have ruled out so far by our normalization. But then the off-set function, which is $h(\vec{\theta}) = 2(-1)^{N-1}(\cos N\theta_1 + \cos N\theta_2)$ does not satisfy the neccessary conditions formulated in Lemma **??**. But since

$$h^{-1}(0) = \{(\theta_1,\theta_2)|\theta_1 \pm \theta_2 = \pi\}$$

a linear transformation performed on the Bloch-parameters allows us to obtain the form neccessary for the lemma, if uv or uv^{-1} play the rôle of the operator of multiplication by z. To make that point more precise, let us denote the Weyl-Schrödinger representation by $(\rho_\theta, L^2(S^1))$ and consider the automorphism η of $\mathcal{A}_\gamma$ given by $\eta(u) = v^{-1}u$, $\eta(v) = v$. Lemma **??** can be easily adapted to a situation where one considers polynomial solutions in the representation $(\rho_\theta \circ \eta, L^2(S^1))$. In fact, one just has to replace h by $h\circ\hat{\eta}$ where $\hat{\eta}$ is the transformation on the Bloch-parameters induced by η. And indeed, $\rho_\theta\circ\eta(H(a,b,0,0))$ preserves $\mathcal{P}_N$, for $\frac{\gamma}{2\pi} = \frac{M}{N}$ if $a = b$ and $\theta = \frac{\pi}{2N}$. The resulting Bethe ansatz-equations and energies are straightforwardly obtained, they may also be found in [**?**], the energies are also "middle points". In this representation too, analytic Bethe ansatz-solutions cannot be found.

4.3.2 *Quantum pendulum-integral at $k = 1$*

The operator $H(b,b,k)$ will be called quantum pendulum-integral, or short QP-integral, as it is derived from an integral of motion of the (discrete) quantum pendulum; this will be explained in Section 5. Let us concentrate here on the family

$$H_{QP}(n,k) := H(2\cos\frac{n}{2}\gamma, 2\cos\frac{n}{2}\gamma, k)$$

and specialize the results of Corollary 2 to the case $k = 1$, $n \in \mathbb{N}$. That case is special in that the critical circle of the off-set function which corresponds to a

Bethe ansatz-solution is in fact an absolute extremum. In fact, by (**??**),

$$h(2\cos\frac{n}{2}\gamma, 2\cos\frac{n}{2}\gamma, 1; \pi + \frac{n-1}{2}, \theta_2) = 4(-1)^{M-1}$$

which is for M even (odd) the absolute minimum (maximum). The Bethe ansatz-eigenvalues are thus at h-band edges, but lateron we shall observe (but not prove) that they describe h-band touching for all but finite rational values of the flux.

The Bethe ansatz-equations for the z_m are given by

$$\frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}} + q^{\frac{1}{2}} z_j^{-1} + q^{-\frac{1}{2}} z_j}{q^{\frac{n}{2}} + q^{-\frac{n}{2}} + q^{-\frac{1}{2}} z_j^{-1} + q^{\frac{1}{2}} z_j} = -\prod_{i=1}^{n-1} \frac{q z_j - z_i}{z_j - q z_i}. \tag{4.22}$$

The energies become

$$E(z_1, \dots, z_{n-1}) = -4\cos\frac{n}{2}\gamma\cos\frac{n-1}{2}\gamma - 4\sin\frac{\gamma}{2}\sin\frac{n-1}{2}\gamma\sum_{j=1}^{n-1} z_j. \tag{4.23}$$

The first solutions are easy to obtain. But already for $n = 3$ we have to employ the matrix representation (**??**) to obtain the eigenvalues. In that case they are given by

$$\begin{aligned} E_1 &= -4\cos\frac{3}{2}\gamma\cos\gamma \\ E_{2\pm} &= -2(\cos\gamma + 1)\cos\frac{3}{2}\gamma \pm 2\sqrt{(\cos\gamma - 1)^2\cos^2\frac{3}{2}\gamma + 8\sin^2\frac{\gamma}{2}\sin^2\gamma}. \end{aligned}$$

They are depicted in Figure **??**.

4.4 Relating the Hofstadter Hamiltonian to the QP-integral

The "middle points" in the spectrum of the isotropic Hofstadter Hamiltonian have been obtained upon application of Corollary **??**. Corollary **??** furnished us with energy values, in particular of the QP-integrals at $k = 1$, which depend analytically on the flux. We will now show that these two cases are interrelated, and explain how the Bethe ansatz-solution for the QP-integrals contain the "middle points" in the spectrum of the Hofstadter. In fact, there are two ways to relate the Hofstadter Hamiltonian with QP-integrals, and we will discuss both of them.

But before we do so, note, that to $H(a, 2\cos\frac{n\gamma}{2}, 1)$, and hence to the QP-integral $H_{QP}(n, 1)$, both, Corollary **??** and **??** apply at $\theta = \pi + \frac{n-1}{2}\gamma$ for rational flux. Thus if $\frac{\gamma}{2\pi} = \frac{M}{N}$, where $(M, N) = 1$ and $N > n$, then $H(a, 2\cos\frac{n\gamma}{2}, 1)$ preserves $\mathcal{P}_n$ and $\mathcal{P}_{n,N-1}$ in the Schrödinger representation with the above angle. In that case, the Bethe ansatz-ansatz equations (**??**) determine eigenvectors and -values belonging to $\mathcal{P}_{n,N-1}$ and equations (**??**) eigenvectors and -values belonging to $\mathcal{P}_n$. Since the eigenvectors belonging to $\mathcal{P}_{n,N-1}$ are divisible by z^n, all solutions of (**??**) are of a form where $z_1, \dots, z_n = 0$ (up to a permutation of the indices).

Case 1. Let us consider first the QP-integral $H_{QP}(n,k)$, i.e. $a = b = 2\cos\frac{n\gamma}{2}$, at rational flux $\frac{\gamma}{2\pi} = \frac{M}{2n}$ (M odd and coprime to n). Then $a = b = 0$ so that for this value of the flux $k^{-1}H_{QP}(n,k)$ coincides with the anisotropic Hofstadter Hamiltonian at flux $2\gamma = 2\pi\frac{M}{n}$. This has an immediate consequence for the special case of $k = 1$.

Theorem 5 *For $\frac{\gamma}{2\pi} = \frac{M}{2n}$ with $(M, 2n) = 0$, the Bethe ansatz-eigenvalues of $H_{QP}(n,1)$ acting in the Schrödinger representation with $\theta = \pi + \frac{n-1}{2}\gamma$ are points where h-bands touch and coincide with the "middle points" in the spectrum of the isotropic Hofstadter Hamiltonian at flux $2\pi\frac{M}{n}$.*

Proof: Recall from above that under the hypothesis of the theorem both corollaries apply to $H_{QP}(n,1)$ and that therefore equations (**??**) have only solutions for which at least n of the variables z_j vanish. Upon inserting $z_1, \dots, z_n = 0$ into (**??**) which now becomes (**??**) one obtains precisely (**??**), the term $q^{\frac{n}{2}} - q^{-\frac{n}{2}}$ vanishing for the above flux. Using $\theta_b = \frac{n}{2}\gamma = \frac{\pi M}{2}$ and taking into account that for the above value of γ also $\sin\frac{n-1}{2}\gamma = \sin\frac{\pi M}{2}\cos\frac{\gamma}{2}$ one sees that the formulas for the energies, (**??**) and (**??**), which now become (**??**) and (**??**), coincide. Hence we may conclude that each eigenvalue of $H_{QP}(n,1)$ is in the representation of the theorem two fold degenerate[5] and is a "middle point" in the spectrum of the isotropic Hofstadter Hamiltonian. This implies that the eigenvalues – of which we know that they lie at h-band edges – actually are points where h-bands touch. q.e.d.

Note that the last result is in perfect agreement with the observation that, with γ as in the theorem, the Hofstadter Hamiltonian has at flux 2γ at most n bands but $H_{QP}(n,1)$, at flux γ, $2n$ h-bands.

Case 2. We have seen in the last paragraph that the Bethe ansatz-solutions of the family of QP-integrals contain information on the spectrum of the Hofstadter Hamiltonian at rational flux. Surprisingly, QP-integrals are also related to the square of the Hofstadter Hamiltonian with half flux. In fact, setting $\tilde{\gamma} = \frac{\gamma}{2}$ and upon substituting $u = q^{-\frac{1}{4}}\breve{u}\breve{v}^*$ and $v = q^{\frac{1}{4}}\breve{u}\breve{v}$ one obtains

$$kH_{Hof}^2(k, \frac{\gamma}{2}) - 2(k + k^{-1}) = H(2\cos\frac{\gamma}{4}, 2\cos\frac{\gamma}{4}, k) = H_{QP}(\frac{1}{2}, k).$$

Thus the square of the Hofstadter Hamiltonian for flux $\frac{\gamma}{2}$ is up to multiplicative and constants equal to the QP-integral at $b = 2\cos\frac{\gamma}{4}$ for flux γ. Specifying to $k = 1$, $b = 2\cos\frac{\gamma}{4}$ is not a value for which Corollary **??** applies but if $\gamma = 2\pi\frac{2M'}{2n+1}$, $(M', 2n+1) = 1$ then $2\cos\frac{n\gamma}{2} = (-1)^{M'}2\cos\frac{\gamma}{4}$, and hence, in the Schrödinger representation with angle $\theta = \pi + \frac{n-1}{2}\gamma$, the Bethe ansatz-eigenvalues of $H_{QP}(n,1)$ are as well eigenvalues for the square of the Hofstadter

[5]Recall that the Weyl-Schrödinger representation provides the first step of the Bloch-decomposition for the operators which we consider. This implies that each (generalized) eigenvalue can be at most be twofold degenerate.

Hamiltonian for flux $\frac{\gamma}{2} = 2\pi\frac{M'}{2n+1}$. The relation between the different Bethe ansatz-equations is not so clear in this case but one can show that the eigenvalue of $H^2_{Hof}(1)$ which is not obtained as a Bethe ansatz-eigenvalue of $H_{QP}(n,1)$ is the one that at $E = 0$. Taking into account that $H^2_{Hof}(k)$ has for flux $2\pi\frac{M'}{2n+1}$ only $n+1$ bands (the spectrum of $H_{Hof}(k)$ is mirror symmetric around 0) it follows that n h-bands of $H_{QP}(n,k)$ have to touch for $\gamma = 2\pi\frac{2M'}{2n+1}$. There is strong evidence that (**??**) describes again the n points at which this h-band touching occurs.

5 Discrete Sine Gordon Field Theory

In this subsection we summarize some notions and facts about the doubly discrete sine-Gordon theory as much as it is necessary for the remainder of this article. For details we refer to the articles by Bobenko and Pinkall [**?**] and of Faddeev and Volkov [**?**] in the first part of this book and - of course - to the original literature as e.g. [**?**],[**?**],[**?**],[**?**],[**?**],[**?**], [**?**],[**?**],[**?**].

The doubly discrete sine-Gordon model is a particularly interesting example of a field theory on a two dimensional space-time with Lorentz metric. Dynamics is Einstein causal and integrable in the classical as well as in the quantum sense. The classical model possesses a nice geometric interpretation which parallels the continuum theory to a surprising extent [**?**] and which provides a most usefull guiding principle for the analysis of the model. This geometric picture suggests the interpretation of the sine-Gordon field as an observable field in the sense of algebraic quantum field theory, living on the faces of a discrete surface in $\mathbb{R}^3$ with constant negative Gaussian curvature, and to look at it as the gauge invariant part of a larger field algebra, living on the edges of the discrete surface.

The usefullness of this point of view becomes evident in the derivation of quantum integrals by the inverse scattering method and in the determination of eigenvalues and eigenvectors for some of them. In both cases, operators play a role, which are not in the observable algebra, i.e. in the algebra generated by the sine-Gordon field. In particular, the ladder operator used in the Bethe ansatz does not belong to this algebra.

5.1 Sine-Gordon theory as gauge theory

The key observation in the analysis of the discrete sine-Gordon model from a geometric point of view is due to Bobenko and Pinkall: It is the surprising fact, that all concepts of continuum theory of surfaces of constant negative curvature can be taken over to the discrete setting, if the key geometrical concepts like that of constant negative curvature, Gauss map etc. are properly chosen. A two dimensional discrete surface with constant negative Gaussian curvature, or simply a (discrete) K-surface, is defined by a map $F : \mathbb{Z}^2 \to \mathbb{R}^3$ satisfying certain conditions reflecting the geometrical properties. It is convenient to view $\mathbb{Z}^2$ as the set of vertices in a light cone net $\mathcal{L}$. By this we mean that its standard basis $\{e_r, e_l\}$ is interpreted in such a way that e_l points into the left moving light cone and e_r into the right moving one in two dimensional Minkowski space. Furthermore, two neighboured lattice points are joint by oriented edges. More precisely, we denote by $\epsilon_l(n)$ and $\epsilon_r(n)$ the edges which have source n and range $n_l := n + e_l$ and $n_r := n + e_r$, respectively. The obvious squares, each one being surrounded by four edges, will be referred to as faces. Analogously, edges and faces in the discrete surface, the image of F, are defined. A K-surface is up to an overall translation and scaling uniquely determined by its so-called Gauss map $N : \mathbb{Z}^2 \to S^2$ [**?**]. $N(n)$ is the normal[6] to the surface at $F(n)$. We like to view N as S^2-valued vertex field, which we call the normal field. An arbitrary normal field configuration provides the Gauss map of a K-surface if it satisfies the discrete field equation

$$\Box N := N(n) + N(n_u) - N(n_l) - N(n_r) = \rho(n)(N(n) + N(n_u) + N(n_l) + N(n_r)) \tag{5.1}$$

$\rho : \mathbb{Z}^2 \to \mathbb{R}$ being a scalar field determined by the equation and $n_u = n + e_l + e_r$. The normal field subject to this equation is the discrete version of the $O(3)$-invariant chiral model [**?**]. N is closely related, but not equal to the sine Gordon field, and for various reasons it is not the field which will be quantized.

As often in physics, it turns out to be fruitful to consider fields with more internal degrees of freedom describing the original one with the help of a gauge group. In this spirit we consider not only the Gauss map but a whole moving frame $\Psi : \mathbb{Z}^2 \to SU(2)$. Using an orthonormal basis $\{f_i\}_{i=1,2,3}$ for $\mathbb{R}^3$, $\Psi(n)$, acting (from the right) in the spin 1 representation, describes the dreibein $\{\Psi(n)^{-1} \cdot f_i\}_{i=1,2,3}$ attached at $F(n)$ with 3-direction pointing towards the normal, hence $N(n) = \Psi(n)^{-1} \cdot f_3$. Taking (multiplicative) differences along edges defines an edge field V,

$$V(\epsilon) := \Psi(r(\epsilon))\Psi(s(\epsilon))^{-1},$$

denoting by ϵ an edge and by $s(\epsilon)$ and $r(\epsilon)$ its source and range, respectively. The edge field has to satisfy the compatibility condition that for any four edges $\epsilon_1, \cdots, \epsilon_4$ forming a closed path, that is for which $r(\epsilon_i) = s(\epsilon_{i+1})$, $i \in \mathbb{Z}_4$, holds

$$V(\epsilon_4)V(\epsilon_3)V(\epsilon_2)V(\epsilon_1) = 1. \tag{5.2}$$

In fact this is a discrete version of an $SU(2)$-bundle over $\mathbb{Z}^2$ and its associated spin one vectorbundle. V takes values in $SU(2)$, but an arbitrary edge field configuration, even if it satisfies (**??**), does not always give rise to a moving frame of a K-surface. In particular, one has to incorporate a consequence of the field equation for N, namely that the angle $\Delta_r(n)$ between $N(n)$ and $N(n_r)$ does not depend on n_2 (that is, is constant in left moving direction) and the angle $\Delta_l(n)$ between $N(n)$ and $N(n_l)$ not on n_1. In fact, for the present discussion of discrete sine Gordon theory one restricts to the case in which these two angles

[6] As part of the definition of K-surfaces, edges emanating from a point $F(n)$ lie in one plane. The normal at $F(n)$ is defined by this plane.

are constant along the whole surface [7]. We denote them by Δ_l, Δ_r. In that case V has, in its fundamental representation, the form

$$V(\epsilon) = \cos\frac{\Delta(\epsilon)}{2} \begin{pmatrix} e^{i\alpha(\epsilon)} & i\tan\frac{\Delta(\epsilon)}{2} e^{i\beta(\epsilon)} \\ i\tan\frac{\Delta(\epsilon)}{2} e^{-i\beta(\epsilon)} & e^{-i\alpha(\epsilon)} \end{pmatrix} \tag{5.3}$$

with $\alpha(\epsilon), \beta(\epsilon) \in S^1$ and $\Delta(\epsilon) = \Delta_{l,r}$ if ϵ is parallel to $e_{l,r}$. Thus in the present situation the edge field takes values in a two-dimensional submanifold of $SU(2)$ which is homeomorphic to the 2-torus $S^1 \times S^1$ (but it is not a group). This $S^1 \times S^1$-valued edge field $\epsilon \mapsto (\alpha(\epsilon), \beta(\epsilon))$ plays an important role. It is this field which shall lateron be quantized. Keeping in mind that the Gauss-map already determines the surface, it is natural to consider a change of the moving frame Ψ by (pointwise) multiplication with a rotation leaving the normal of every point $F(n)$ fixed as gauge transformation. The gauge group is thus given by $U(1)$-valued vertex fields $G : \mathbb{Z}^2 \to U(1) \subset SU(2)$, with $U(1)$ being the subgroup which stabilizes f_3 in the spin 1 representation. G acts on edge fields by conjugation

$$V(\epsilon) \mapsto G(r(\epsilon))V(\epsilon)G(s(\epsilon))^{-1}.$$

This gauge transformation may as well be formulated as a gauge transformation on the angles $\alpha(\epsilon)$ and $\beta(\epsilon)$ entering in (**??**) separately.

A change of the angles Δ_l, Δ_r gives rise to a different Gauss map and hence a different surface. But if one changes the angles in such a way that the product

$$k := \tan\frac{\Delta_l}{2}\tan\frac{\Delta_r}{2}$$

is left unchanged one obtains a family of surfaces which have equal second fundamental form. One may parametrize this family by a so-called spectral parameter and two functions $\Delta_r, \Delta_l : \mathbb{R}^{>0} \to S^1$ determined by

$$\tan\frac{\Delta_r(\eta)}{2} = \eta^{\frac{1}{2}} k^{\frac{1}{2}}, \quad \tan\frac{\Delta_l(\eta)}{2} = \eta^{-\frac{1}{2}} k^{\frac{1}{2}}.$$

The sine Gordon field w can now be obtained as a gauge invariant expression of the angles $\alpha(\epsilon)$ and $\beta(\epsilon)$ entering in the edge field. It may be interpreted as S^1-valued field living on the faces. If $f(n)$ is the face which has vertices n, n_l, n_r, and $n_u = n + e_l + e_r$ then

$$w(f(n)) := -\alpha(\epsilon_l(n)) + \beta(\epsilon_l(n)) + \alpha(\epsilon_r(n)) - \beta(\epsilon_r(n)).$$

$w(f(n)) + \pi$ can also be expresses as the angle between the vectors $F(n_l) - F(n)$ and $F(n_r) - F(n)$. In particular, the sine Gordon field is not only gauge invariant like the normal field but unlike the latter it is also invariant under

[7] In the continous case this is in fact no restriction [?]. Discrete surfaces have more structure, which we restrict by this ad hoc assumption.

a change of the spectral parameter η. This latter invariance is crucial for the integrability of its field equation. This field equation may be derived from the field equation of the normal field (**??**). It may, however, as well be deduced from the compatibility condition (**??**) assumed to hold for an edge field depending on a spectral parameter through the angles $\Delta_r(\eta), \Delta_l(\eta)$:

$$V(\eta,\epsilon) = \cos\frac{\Delta(\eta,\epsilon)}{2}\begin{pmatrix} e^{i\alpha(\epsilon)} & i\eta^{-\sigma(\epsilon)}k^{\frac{1}{2}}e^{i\beta(\epsilon)} \\ i\eta^{-\sigma(\epsilon)}k^{\frac{1}{2}}e^{-i\beta(\epsilon)} & e^{-i\alpha(\epsilon)} \end{pmatrix}$$

where

$$\sigma(\epsilon) = \begin{cases} \frac{1}{2} & \text{if } \epsilon \text{ is parallel to } e_l \\ -\frac{1}{2} & \text{if } \epsilon \text{ is parallel to } e_r \end{cases}.$$

Writing shorter $w(n)$ for $w(f(n))$ the field equation so obtained, the so-called discrete sine Gordon equation, is given by

$$\Box w(n) \;=\; 2\arg(1 + k\; e^{-iw(n_r)}) \;+\; 2\arg(1 + k\; e^{-iw(n_l)}). \tag{5.4}$$

Equation (**??**) can be rewritten in terms of the coordinates $Q(n) := e^{-iw(n)}$,

$$Q(n)\; Q(n_u) \;=\; M\big(Q(n_l)\big)\; M\big(Q(n_r)\big) \tag{5.5}$$

$$M(z) := \frac{k+z}{1+kz} \;=\; \frac{1}{M(1/z)}.$$

The Möbius transformation M maps the circle to itself.

5.2 Classical dynamics and Poisson structure for the sine Gordon field

For the purpose of quantization we need a formulation of classical sine-Gordon theory in phase space with Poisson structure which is compatible with the dynamics. The time evolution of the classical sine Gordon field is described by (**??**). This is a second order hyperbolic difference equation. A solution of this equation contains therefore the time evolution of the Cauchy-Data along the "Cauchy-zigzag" at time t

$$C_t := \{n \in \mathbb{Z}^2 | n_1 - n_2 = t \text{ or } n_1 - n_2 = t+1\}.$$

In other words, the field configuration on such a zigzag corresponds to initial conditions for the second order difference equation (**??**), because a zigzag consists of two consecutive time slices. Covariant phase space can be identified with the space of field configurations on C_0. It is an infinite dimensional torus $M = (S^1 \times S^1)^{\mathbb{Z}}$, an element (z_1, z_2) is given by $z_1(s) = Q(s(e_r - e_l))$, $z_2(s) = Q(s(e_r - e_l) + e_r)$. The doubly discrete sine–Gordon equation (**??**) induces on phase space a time evolution defined by the shift

$$\alpha : M \;\rightarrow\; M \tag{5.6}$$

$$z \;=\; (z_1, z_2) \;\mapsto\; \alpha(z) \;=\; (z_2, z_3)$$

where $z_3(s) := e^{-iw(s(e_r-e_l)+e_r+e_l)}$. In terms of these coordinates time evolution is

$$z_1(s)\ z_3(s) \ = \ F(s, z_2)\ ,\ (s\ \in\ \mathbb{Z}\ ,\ z_2\ \in\ (S^1)^{\mathbb{Z}}),$$

where F is given by

$$F(s,z) \ = \ \left(\frac{k+z(s)}{1+kz(s)}\right)\ \left(\frac{k+z(s-1)}{1+kz(s-1)}\right). \tag{5.7}$$

The geometric background of the model suggests a natural Poisson bracket on sufficiently regular functions on phase space.[8] It suffices to define the Poisson brackets on coordinate functions only:

$$\left\{w(n), w(n_r)\right\} \ = \ \left\{w(n), w(n_l)\right\} \ = \ 1, \tag{5.8}$$

for all $n = (k,-k), k \in \mathbb{Z}$. All other brackets vanish. It is straightforward to prove, that the equation of motion (**??**) and the Poisson structure (**??**) are compatible (see also the article in this book by Kutz [**?**] and [**?**]) and imply the following Poisson brackets for the discrete classical field w:

$$\left\{w(n), w(m)\right\} \ = \ 0\ ,\ \text{ for } n, m\ \in \mathbb{Z}^2 \text{ space–like}$$

and

$$\left\{w(n), w(n_r)\right\} \ = \ \left\{w(n), w(n_l)\right\} \ = \ 1\ , (n\ \in \mathbb{Z}^2).$$

(n, m space-like means that the Minkowski length of $n - m$ be smaller than 0.) Hence, the theory possesses discrete Einstein causality. Furthermore time evolution is an automorphism of the Poisson algebra of functions on phase space. Let us also mention that the Poisson bracket is compatible with the choice of periodic boundary conditions.

The Poisson structure just introduced, is close, but not the same, as the one of the discrete elastic chain. In fact, the coordinate functions $\tilde{w}(n) \ = \ (-1)^{n_2} w(n)$ have the same Poisson brackets as the relative coordinates of the elastic chain, i.e.

$$\left\{\tilde{w}(n_l), \tilde{w}(n)\right\} \ = \ \left\{\tilde{w}(n), \tilde{w}(n_r)\right\} \ = \ 1 \tag{5.9}$$

for all $n = (k,-k), k \in \mathbb{Z}$. In terms of $\tilde{Q}(n) := e^{-i\tilde{w}(n)}$ time evolution (**??**) is given by

$$\frac{\tilde{Q}(n_u)}{\tilde{Q}(n)} \ = \ \frac{M(\tilde{Q}(n_r))}{M(\tilde{Q}(n_l))}. \tag{5.10}$$

The reformulation of the doubly discrete sine–Gordon model in terms of $\tilde{w}$ and $\tilde{Q} = e^{-i\tilde{w}}$ is called the discrete Volterra model.

[8] To be on the safe side, consider smooth functions, which depend on finitely many arguments only.

5.3 Quantization

It is possible to quantize the sine Gordon field directly in the canonical way, i.e. upon replacing the Poisson bracket (**??**) by canonical commutation relations. But for the reasons which we have already indicated the edge field needs to be quantized as well. Its quantization is guided by the constraint that it is compatible with the quantization of the sine Gordon field. One way to proceed is to first define a Poisson bracket for the edge fields and then to quantize this bracket in a standard manner. This turns out to be the same as replacing the 2-torus in which the edge field take its values by its non commutative analog, the quantized torus. In other words, quantizing the edge field amounts to replacing the two complex numbers $e^{i\alpha(\epsilon)}, ie^{-i\beta(\epsilon)}$ in $V(\epsilon)$ by two unitary operators $u(\epsilon), v(\epsilon)$ subject to the relation $u(\epsilon)v(\epsilon) = e^{-i\frac{\gamma}{2}}v(\epsilon)u(\epsilon)$, the angle[9] γ being proportional to Planck's constant.

5.4 Quantum dynamics of the sine Gordon field

As mentioned before, the classical sine Gordon equation is a consequence of the compatibility equation. One may now define a quantized compatibility equation as being formally the same as (**??**) but with noncommutative entries and derive from it the quantized sine Gordon equation.

For that purpose it turns out to be more convenient to choose the reformulation of the sine Gordon model as Volterra model. Geometrically the change of coordinates $Q(n) \mapsto Q(n)^{(-1)^{n_2}}$ can be achieved by turning the corresponding dreibein by an angle of π around the f_1-axis whenever n_2 is an odd number. The effect on the edge field is easily determined and if we renormalize the edge-field by dropping irrelevant constants $\cos\frac{\Delta_{r,l}(\eta)}{2}$ in front of the matrices, we arrive at the basic equation for the derivation of the (quantum) time evolution. This is the (quantum) compatibility equation

$$\mathcal{V}(\eta, \epsilon_r(n))\mathcal{V}(\eta, \bar{\epsilon}_l(n)) = \mathcal{V}(\eta, \bar{\epsilon}_l(n_r))\mathcal{V}(\eta, \epsilon_r(n_l)) \tag{5.11}$$

where

$$\mathcal{V}(\eta, \epsilon) = \begin{pmatrix} u(\epsilon) & -\eta^{\frac{1}{2}}k^{-\sigma(\epsilon)}v^*(\epsilon) \\ \eta^{\frac{1}{2}}k^{-\sigma(\epsilon)}v(\epsilon) & u^*(\epsilon) \end{pmatrix} \tag{5.12}$$

and $\bar{\epsilon}$ is edge ϵ with reversed orientation. $\mathcal{V}(\eta, \epsilon)$ comes also under the name Volterra L-matrix. The strategy is to look for an automorphism $\varphi \in \mathrm{Aut}(\mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}})$ which, when extended to an automorphism $\hat{\varphi} \in \mathrm{Aut}(\mathrm{Mat}_2(\mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}}))$ by entrywise action, becomes "twisted half time evolution" of the edge field along a face. Mathematically this means that

$$\mathcal{V}(\eta, \bar{\epsilon}_l(n_r))\mathcal{V}(\eta, \epsilon_r(n_l)) = \hat{\varphi}(\mathcal{V}(\eta, \bar{\epsilon}_l(n))\mathcal{V}(\eta, \epsilon_r(n))). \tag{5.13}$$

One then extends $\hat{\varphi}$ to a whole zigzag and combines this extension with the extension conjugated by a discrete space translation to obtain the discrete time evolution.

[9]The factor $\frac{1}{2}$ in front of the angle is chosen for better comparison of formulas lateron.

To explain this in detail we abandon the geometric "space-time" indexing in favour of a purely "space" indexing which is better adapted to the algebraic aspects. The reason for this notational change is that first, from now on everything takes place in one time slice only, i.e. on edges $\epsilon_l(n), \epsilon_r(n)$ with constant $n_1 - n_2$, and second, the product above (**??**) is the product of 2×2 matrices in which the entries multiply with the tensor product. We therefore use consistently the following notation: For $X, Y \in \mathrm{Mat}_n(\mathcal{A})$ the product $X_2 Y_1$ shall be the matrix $Z \in \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A})$ which has entries $Z_{ij} = \sum_l X_{il} \otimes Y_{lj}$, and similarily are products like $Z_3 X_2 Y_1$ etc. defined. This notation is now adapted to the space structure of the time slice at $t = 0$ by identifying $\mathcal{V}^-_{2s-1}(\eta)$ with $\mathcal{V}(\eta, \bar{\epsilon}_l(s, -s))$ and $\mathcal{V}^+_{2s}(\eta)$ with $\mathcal{V}(\eta, \epsilon_r(s, -s))$, where

$$\mathcal{V}^{\pm}_s(\eta) = \begin{pmatrix} u_s & -\eta^{\frac{1}{2}} k^{\pm\frac{1}{2}} v_s^* \\ \eta^{\frac{1}{2}} k^{\pm\frac{1}{2}} v_s & u_s^* \end{pmatrix}, \tag{5.14}$$

and we recall from above that $u_s v_s = e^{-i\frac{\gamma}{2}} v_s u_s$. In the new notation (**??**) reads

$$\mathcal{V}^+_2(\eta)\mathcal{V}^-_1(\eta) = \hat{\varphi}(\mathcal{V}^-_2(\eta)\mathcal{V}^+_1(\eta)). \tag{5.15}$$

Condition (**??**) does not uniquely determine the action of φ, very much like the classical compatibility equation does not determine a unique time evolution of the edge field. We shall now derive the restriction of φ to the largest subalgebra of $\mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}}$ on which the action of φ is infact uniquely determined by (**??**). Let us set

$$L(\eta, k) := \mathcal{V}^+_2(\eta)\mathcal{V}^-_1(\eta)$$

where we indicate for a moment with the dependence on the parameter k, that condition (**??**) may be reformulated as

$$\hat{\varphi}(L(\eta, k^{-1})) = L(\eta, k), \tag{5.16}$$

for all η. $L(\eta, k)$ is an element of $\mathrm{Mat}_2(\mathcal{S})$ where $\mathcal{S}$ is the subalgebra $\mathcal{S} \subset \mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}}$ generated by the unitaries

$$U = u_2^{-1} u_1^{-1}, \quad V = u_2 v_1^{-1}, \quad Z = u_2 v_2^{-1} u_1 v_1 \tag{5.17}$$

which are subject to the relation $UV = q^{-\frac{1}{2}} VU$ and Z is a central element. Hence,

$$\mathcal{S} \cong \mathcal{A}_{\frac{\gamma}{2}} \otimes C(S^1).$$

For irrational $\frac{\gamma}{2\pi}$, $\mathcal{S}$ is the fixed point algebra of $\mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}}$ under the automorphism given by conjugation with Z.

Proposition 3 *Let φ be an automorphism of $\mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}}$ whose extension $\hat{\varphi}$ satisfies (**??**). Then its restriction to $\mathcal{S} \subset \mathcal{A}_{\frac{\gamma}{2}} \otimes \mathcal{A}_{\frac{\gamma}{2}}$ is given by*

$$\varphi(U) = U \tag{5.18}$$

$$\varphi(Z) = Z \tag{5.19}$$

$$\varphi(V) = V \frac{h(k)}{h(k^{-1})} =: V f(U^2 Z) \tag{5.20}$$

where $h(k) = k^{-\frac{1}{2}} + k^{\frac{1}{2}} q^{-\frac{1}{2}} U^2 Z$, *i.e.*

$$f(z) = \frac{k^{-\frac{1}{2}} + k^{\frac{1}{2}} q^{-\frac{1}{2}} z}{k^{\frac{1}{2}} + k^{-\frac{1}{2}} q^{-\frac{1}{2}} z}. \tag{5.21}$$

Proof: In the variables of $\mathcal{S}$, $L(\eta, k)$ has the form

$$L(\eta, k) = \begin{pmatrix} U^{-1} - \eta U Z & -\eta^{\frac{1}{2}} V h(k) \\ \eta^{\frac{1}{2}} (V h(k))^* & U - \eta U^{-1} Z^{-1} \end{pmatrix}.$$

Since (**??**) has to hold for all η, (**??**,**??**) follow immediately and

$$\varphi(V h(k^{-1})) = \varphi(V) h(k^{-1})$$

which has to be equal to $V h(k)$. From that relation follows (**??**). This extends in fact to an automorphism on $\mathcal{S}$ since $\frac{h(k)}{h(k^{-1})}$, which we define to be 1 in case $k = 1$, is unitary and commutes with U and V. q.e.d.

Given a natural number p let $\check{\varphi} = \varphi^{\otimes 2p}$ be the extension of φ to the algebra $\mathcal{A}_{\frac{\gamma}{2}}^{\otimes 2p}$. In the spirit of algebraic quantum field theory we like to view $\mathcal{A}_{\frac{\gamma}{2}}^{\otimes 2p}$ as the field algebra on a fixed time slice consisting of $2p$ points. We consider space as periodic so that the automorphism $\sigma \in \mathrm{Aut}(\mathcal{A}_{\frac{\gamma}{2}}^{\otimes 2p})$ given by the cyclic shift in the tensor product, i.e. by $\sigma(a_n) = a_{n+1}$, plays the rôle of translation in space.

Definition 2 *The (discrete) time evolution is given by an automorphism*

$$\alpha = \check{\varphi} \circ \sigma \circ \check{\varphi} \circ \sigma^{-1}$$

of $\mathcal{A}_{\frac{\gamma}{2}}^{\otimes 2p}$ *where the extension* $\hat{\varphi}$ *of* $\varphi \in \mathrm{Aut}(\mathcal{A}_{\frac{\gamma}{2}}^{\otimes 2})$ *satisfies (**??**).*

The automorphism α is also refered to as time 1 map. As already indicated, it cannot be uniquely determined by condition (**??**). We will show now that its restriction to the subalgebra generated by the elements $C_n := u_n^{-1} u_{n-1}^{-1} v_n^{-1} v_{n-1}$, $n = 1, \ldots, 2p \pmod{2p}$ is uniquely determined by this condition. In fact, this subalgebra forms in the classical case the algebra of gauge invariant quantities on which time evolution given by the discrete sine-Gordon equation is defined.

Lemma 3 *Suppose that the extension* $\hat{\varphi}$ *of* $\varphi \in \mathrm{Aut}(\mathcal{A}_{\frac{\gamma}{2}}^{\otimes 2})$ *satisfies (**??**). Let* $\check{\varphi}_{\pm} := \check{\varphi} \circ \sigma^{\pm 1}$ *and* $C_n := u_n^{-1} u_{n-1}^{-1} v_n^{-1} v_{n-1}$. *Then*

$$\check{\varphi}_{\pm}(C_{2n+1}) = C_{2n+1 \pm 1}.$$

Proof: First observe, that

$$C_{2n} = U_n^2 Z_n \tag{5.22}$$

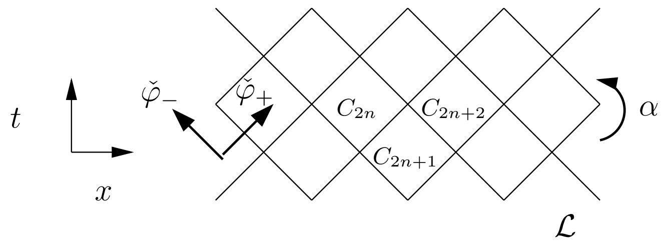
$$C_{2n+1} = U_n^{-1} Z_n^{-1} V_n^{-1} V_{n+1} U_{n+1} \tag{5.23}$$

due to (**??**). Therefore, $\check{\varphi}$ acts trivially on half of the C_n, namely

$$\check{\varphi}(C_{2n}) = C_{2n}.$$

This implies the statement. q.e.d.

The last lemma is not only of technical value, but it also helps visualizing the time evolution which we have sketched in the figure below:



The generators C_n with even index are invariant under $\check{\varphi}$ and those with odd index are invariant under $\check{\varphi}_\pm \circ \check{\varphi} \circ \check{\varphi}_\pm^{-1}$. The equations satisfied by $\check{\varphi}_\pm$ suggest to interprete them as light cone shifts and hence time evolution to be their composition $\alpha = \check{\varphi}_+ \circ \check{\varphi}_-$. This is the content of Definition **??**.

Theorem 6 *Under the conditions of Lemma* **??**

$$\alpha(C_{2n+1}) = f(C_{2n})^{-1} C_{2n+1} f(C_{2n+2}) \tag{5.24}$$
$$\alpha(C_{2n}) = f(\alpha(C_{2n-1}))^{-1} C_{2n} f(\alpha(C_{2n+1})), \tag{5.25}$$

*where f is given by (**??**).*

Proof: Considering generators with odd index we have

$$\begin{aligned} \alpha(C_{2n+1}) &= \check{\varphi}_+(C_{2n}) = \check{\varphi}(C_{2n+1}) \\ &\overset{(??)}{=} \check{\varphi}(U_n^{-1} Z_n^{-1} V_n^{-1} V_{n+1} U_{n+1}) \\ &= U_n^{-1} Z_n^{-1} f(C_{2n})^{-1} V_n^{-1} V_{n+1} f(C_{2n+2}) U_{n+1} \\ &= f(C_{2n})^{-1} C_{2n+1} f(C_{2n+2}), \end{aligned}$$

where we used that $f(x)^{-1} = f(x^{-1})$. The proof for the generators with even index is likewise. q.e.d.

We thus have obtained a local quantum evolution law for all faces of the light cone lattice. In terms of the sine Gordon variables $Q_{2s} = C_{2s}^{(-1)^s}$, $Q_{2s+1} = C_{2s+1}^{(-1)^s}$ the above yields

$$\alpha(Q_{2n+1}) = \frac{k + q^{\frac{1}{2}} Q_{2n}}{1 + q^{\frac{1}{2}} k Q_{2n}} \frac{k + q^{\frac{1}{2}} Q_{2n+2}}{1 + q^{\frac{1}{2}} k Q_{2n+2}} Q_{2n+1}^{-1} \tag{5.26}$$

$$\alpha(Q_{2n}) = \frac{k + q^{\frac{1}{2}} \alpha(Q_{2n-1})}{1 + q^{\frac{1}{2}} k \alpha(Q_{2n-1})} \frac{k + q^{\frac{1}{2}} \alpha(Q_{2n+1})}{1 + q^{\frac{1}{2}} k \alpha(Q_{2n+1})} Q_{2n}^{-1}. \tag{5.27}$$

This is discrete quantum sine-Gordon equation. Equations (**??**) and (**??**) are the quantized form of equation (**??**), with the following correspondence between classical and quantum fields:

$$Q_n \sim Q_{2n+1}, \quad Q_{n_r} \sim Q_{2n+2}, \quad Q_{n_l} \sim Q_{2n}, \quad Q_{n_u} \sim \alpha(Q_{2n+1}).$$

Similarly (**??**) and (**??**) relate to equation (**??**).

5.5 Quantum integrals for periodic boundary conditions

With the help of the inverse scattering method operators may be found which are invariant under the above automorphism. For that one has to impose periodic boundary conditions so that we require p now to be finite. In fact, lateron we shall be interested in the case of $p = 2$.

Denote by $\mathrm{tr}_{\mathbb{C}^2} : \mathrm{Mat}_2(\mathcal{A}) \to \mathcal{A}$ the partial trace given by $\mathrm{tr}_{\mathbb{C}^2}(X) = X_{11} + X_{22}$, where $\mathcal{A}$ as any C^*-algebra.

Lemma 4 *If φ satisfies the conditions of Lemma* **??** *then*

$$\mathcal{F}(\eta) := \mathrm{tr}_{\mathbb{C}^2}(\mathcal{V}^+_{2p}(\eta)\mathcal{V}^-_{2p-1}(\eta)\ldots\mathcal{V}^+_2(\eta)\mathcal{V}^-_1(\eta))$$

is invariant under $\check{\varphi}_\pm$.

Proof: We have, writing shorter $\mathcal{V}^\pm = \mathcal{V}^\pm(\eta)$,

$$\begin{aligned}
\check{\varphi}_+ \,\mathrm{tr}_{\mathbb{C}^2}(\mathcal{V}^+_{2p}\mathcal{V}^-_{2p-1}\ldots\mathcal{V}^+_2\mathcal{V}^-_1) &= \check{\varphi}\,\mathrm{tr}_{\mathbb{C}^2}(\mathcal{V}^+_1\mathcal{V}^-_{2p}\ldots\mathcal{V}^+_3\mathcal{V}^-_2)\\
&= \check{\varphi}\,\mathrm{tr}_{\mathbb{C}^2}(\mathcal{V}^-_{2p}\ldots\mathcal{V}^+_3\mathcal{V}^-_2\mathcal{V}^+_1)\\
&= \mathrm{tr}_{\mathbb{C}^2}(\hat{\varphi}(\mathcal{V}^-_{2p}\mathcal{V}^+_{2p-1})\ldots\hat{\varphi}(\mathcal{V}^-_2\mathcal{V}^+_1))\\
&= \mathrm{tr}_{\mathbb{C}^2}(\mathcal{V}^+_{2p}\mathcal{V}^-_{2p-1}\ldots\mathcal{V}^+_2\mathcal{V}^-_1)
\end{aligned}$$

where we have used the cyclicity of the partial trace $\mathrm{tr}_{\mathbb{C}^2}$ in the second step. Clearly $F(\eta)$ is invariant under a double shift σ^2, hence the assertion follows. q.e.d.

5.6 Quantum integrals of the (discrete) quantum pendulum

We shall now apply the above lemma to the case $p = 2$ which is the smallest nontrivial choice and corresponds to a field theory in $0+1$ dimensions. Since the classical continuous sine-Gordon equation,

$$\omega_{xx} - \omega_{tt} = b \sin\omega \qquad b > 0,$$

becomes the equation of the mathematical pendulum if one looks for x-independent solutions we call that theory the discrete quantum pendulum. The monodromy matrix in the case of $p = 2$ is

$$\mathcal{M}(\eta) := L_2(\eta, k)L_1(\eta, k).$$

It is a 2×2-matrix with values in $\mathcal{A}$ which satisfies $\mathcal{M}_{22}(\eta)=\mathcal{M}_{11}^*(\eta)$ and $\mathcal{M}_{21}(\eta)=-\mathcal{M}_{12}^*(\eta)$ where we treat the spectral parameter as a variable which is invariant under the star operation. By Lemma **??** its trace, which has the form $\mathcal{F}(\eta)=\mathcal{M}_{11}(\eta)+\mathcal{M}_{11}^*(\eta)$, is an integral of motion. The expansion into powers of η of $\mathcal{M}_{11}(\eta)$,

$$\mathcal{M}_{11}(\eta)=A^{(0)}-\eta A^{(1)}+\eta^2A^{(2)} \tag{5.28}$$

results in

$$\begin{aligned}A^{(0)}&=U_2^{-1}U_1^{-1}\\ A^{(1)}&=U_2Z_2U_1^{-1}+U_1Z_1U_2^{-1}+V_2h_2h_1^*V_1^{-1}\\ A^{(2)}&=U_2Z_2U_1Z_1,\end{aligned}$$

where $h_i=h_i(k)$. Hence, for $p=2$ the inverse scattering method provides us with three integrals of motion (quantum integrals). Two of them, $A^{(0)}+{A^{(0)}}^*$ and $A^{(2)}+{A^{(2)}}^*$, turn out not to be very interesting. The remaining one is the one we are looking for. Let us give it here the short name:

Definition 3 *The operator*

$$\tilde{H}=A^{(1)}+{A^{(1)}}^*$$

is called SG-integral.

5.6.1 *The QP-integral*

The SG-integral $\tilde{H}=A^{(1)}+{A^{(1)}}^*$ is an operator which depends only on the unitaries $T=V_1^{-1}V_2$, $C_4=U_2^2Z_2$, $A^{(0)}$ and $A^{(2)}$. The algebra spanned by these operators is a subalgebra of $\mathcal{S}\otimes\mathcal{S}$ in which $A^{(0)}$ and $A^{(2)}$ are central, i.e. $A^{(0)}$ and $A^{(2)}$ commute with C_4 and T and among themselves. The QP-integral is a reduction of the SG-integral which is achieved by imposing a relation among the eigenvalues of $A^{(0)}$ and $A^{(2)}$. Such a reduction is algebraically defined, but at the end of this section we provide a physical interpretation for it.

To describe the reduction we first express the SG-integral in the above mentionned variables.

$$\begin{aligned}A^{(1)}=&A^{(0)}C_4+A^{(2)}C_4^{-1}+q^{\frac12}A^{(0)}{A^{(2)}}^{-1}C_4T\\ &+q^{\frac12}C_4T+k^{-1}T+kqA^{(0)}{A^{(2)}}^{-1}C_4^2T.\end{aligned}$$

Consequently, the SG-integral is given by

$$\begin{aligned}\tilde{H}=&(A^{(0)}+{A^{(2)}}^{-1})C_4+(1+A^{(0)}{A^{(2)}}^{-1})q^{\frac12}C_4T\\ &+k^{-1}T+kqA^{(0)}{A^{(2)}}^{-1}C_4^2T+h.c..\end{aligned}$$

This looks a bit cumbersome, but if we introduce a square root w of $A^{(0)}{A^{(2)}}^{-1}$ and use variables

$$P_1=wC_4 \tag{5.29}$$

$$P_2=q^{\frac12}wC_4T=q^{\frac12}P_1T \tag{5.30}$$

then $P_1P_2 = q^{\frac{1}{2}}w^2C_4^2T$ and $P_1P_2^* = q^{\frac{1}{2}}T^{-1}$ and consequently

$$\tilde{H} = (wA^{(2)} + h.c.)\,P_1 + (w + h.c.)\,P_2 + k^{-1}q^{-\frac{1}{2}}P_1P_2^{-1} + kq^{\frac{1}{2}}P_1P_2 + h.c.. \quad (5.31)$$

Note that P_1 and P_2 span the rotation algebra $\mathcal{A}_\gamma$, namely they are unitaries subject to the relation

$$P_1P_2 = q^{-1}P_2P_1.$$

The algebra $\mathcal{C}$ which is generated by the unitaries $\{P_1, P_2, w, A^{(2)}\}$ is called algebra of sine gordon variables. Of course, C_4 and T are elements of $\mathcal{C}$. The unitaries w and $A^{(2)}$ are central elements in this algebra so that

$$\mathcal{C} \cong \mathcal{A}_\gamma \otimes C(S^1 \times S^1).$$

$\mathcal{C} \cap \mathcal{S} \otimes \mathcal{S}$ is contained in the fixed point algebra of $\mathcal{S} \otimes \mathcal{S}$ with respect to the automorphism given by conjugation with $A^{(0)}$. Moreover, it contains only even elements w.r.t. a grading giving U_i and V_i degree 1 and Z_i degree 0.

We are now in a position to come to the concept of reduction which is appropriate to our situation:

Definition 4 *Given two complex numbers z_1, z_2 of modulus 1, the reduction of the SG-integral defined by these numbers is the image of $\tilde{H}$ under the canonical projection $\mathcal{C} \to \mathcal{C}/\mathcal{J}_{z_1,z_2}$ where $\mathcal{J}_{z_1,z_2}$ is the ideal generated by the two elements $w - z_1$, $A^{(2)} - z_2$. If $z_2 = 1$ we call the reduction a QP-integral.*

Since $\mathcal{C}/\mathcal{J}_{z_1,z_2} \cong \mathcal{A}_\gamma$, the reduction defined by z_1, z_2 can be identified with the operator $H(a, b, k)$ of $\mathcal{A}_\gamma$, where, using the notation of section **??**,

$$a = z_1z_2 + z_1^{-1}z_2^{-1} \quad (5.32)$$

$$b = z_1 + z_1^{-1}. \quad (5.33)$$

A QP-integral is thus a reduction $H(a, b, k)$ for which $a = b \in \mathbb{R}$, $|b| \leq 2$. Note that the spectrum of $H(b, b, k)$ is invariant under a sign change of b.

We have defined reductions of the SG-integral via specification of the values of the central elements in $\mathcal{C}$. Let us now show that such a reduction is preserved under the dynamics, and hence all reductions are integrals of motion. For that we first recall that the automorphism $\varphi \otimes \varphi$ defined by (**??**,**??**,**??**) preserves $\mathcal{C} \cap \mathcal{S} \otimes \mathcal{S}$. In fact, it is easily seen that

$$\varphi \otimes \varphi(A^{(0)}) = A^{(0)} \quad (5.34)$$

$$\varphi \otimes \varphi(A^{(2)}) = A^{(2)} \quad (5.35)$$

$$\varphi \otimes \varphi(C_{2i}) = C_{2i} \quad (5.36)$$

$$\varphi \otimes \varphi(T) = \frac{k^{-1} + q^{\frac{1}{2}}C_4}{1 + k^{-1}q^{\frac{1}{2}}C_4}T\frac{1 + k^{-1}q^{\frac{1}{2}}C_2}{k^{-1} + q^{\frac{1}{2}}C_2} \quad (5.37)$$

a result which leads again to (??) and (??). Next, we show that the cyclic permutation σ preserves $\mathcal{C}$. One obtains

$$\sigma^{-1}(A^{(0)}) = \sigma^{-1}(u_4 u_3 u_2 u_1) = A^{(0)} \tag{5.38}$$

$$\sigma^{-1}(A^{(2)}) = \sigma^{-1}(v_4^{-1} v_3 v_2^{-1} v_1) = {A^{(2)}}^{-1} \tag{5.39}$$

$$\sigma^{-1}(C_4) = u_3^{-1} v_3^{-1} u_2^{-1} v_2 = q^{\frac{1}{2}} {A^{(2)}}^{-1} C_4 T \tag{5.40}$$

and, using $\sigma^{-2}(C_4) = C_2$,

$$\sigma^{-1}(P_1) = P_2 \tag{5.41}$$

$$\sigma^{-1}(P_2) = P_1^{-1}. \tag{5.42}$$

In particular, the restriction of σ^{-1} to the subalgebra generated by the P_i is the Fouriertransform.

Theorem 7 *All reductions of the SG-integral are integrals of motion, i.e. invariant under the automorphism α. The QP-integrals and those for which $z_2 = -1$ are even invariant under the automorphism $\check{\varphi}_\pm$.*

Proof: Since the SG-integral is invariant under $\check{\varphi}_\pm$ we only have to check whether $\alpha = \check{\varphi}_+ \circ \check{\varphi}_-$ preserves $\mathcal{J}_{z_1,z_2} \cap \mathcal{S} \otimes \mathcal{S}$. Formulae (??,??) show that $\mathcal{J}_{z_1,z_2} \cap \mathcal{S} \otimes \mathcal{S}$ is preserved under $\check{\varphi}_\pm$ if and only if $z_2 = \pm 1$. From this the last statement follows. Furthermore these formulae show that $\check{\varphi}^2(w^2) = w^2$ and $\check{\varphi}^2(A^{(2)}) = A^{(2)}$ from which the first statement follows. q.e.d.

In analogy with sector-theory (an algebraic field theoretic notion) we may take the numbers z_1, z_2 which define the reduction of the SG-integral as labels for sectors of the SG-integral. By the last theorem they are invariant under time evolution. The QP-integral is thus an integral of motion of the discrete sine Gordon model with space compactified to one point, in a sector with $z_2 = 1$.

6 Algebraic Bethe Ansatz for the SG-Integral

The algebraic Bethe ansatz [?] is a wellknown ansatz to construct eigenfunctions and eigenvalues for quantum integrable models. We shall apply it first to the SG-integral and then perform the reduction to the QP-integral. Note that our Bethe-ansatz for the SG-integral differs slightly from that known in the literature (see e.g. [?]). Nevertheless this difference is important for performing the reduction to the QP-integral.

Roughly speaking, one may view the construction of Bethe ansatz eigenvectors in analogy to the construction of eigenvectors to J_3 in a highest weight representation of $sl_2(\mathbb{C})$. One starts with a highest weight state, which is called in the Bethe ansatz-context Bethe ansatz-groundstate, although it will have nothing to do with the groundstate of the QP-integral, and which is a nullvector of $\mathcal{M}_{21}(\eta)$. One proceeds to construct "excited" states by applying the "ladder operator" $\mathcal{M}_{12}(\eta)$ one or several times. However, the commutation relation between the trace $\mathcal{M}_{11}(\eta)+\mathcal{M}_{22}(\eta)$ and $\mathcal{M}_{12}(\eta)$ (the Young-Baxter relation) is rather more involved then the commutation relation between J_3 and J_+ so that, only with a special choice of spectral parameters $\eta_1,\cdots,\eta_n$, application of $\mathcal{M}_{12}(\eta_1)\cdots\mathcal{M}_{12}(\eta_n)$ to the Bethe ansatz-groundstate will furnish an eigenvalue of $\mathcal{M}_{11}(\eta)+\mathcal{M}_{22}(\eta)$. The system of equations which determines these spectral values is called Bethe ansatz-equations.

In the present context, where the monodromy matrix was obtained from the edge field, the interpretation of the "excited" Bethe ansatz-states has to be carried out with care. The reason for that is that, whereas the trace of the matrix consists of operators which steem from gauge invariant quantities, this is not the case for the ladder operator $\mathcal{M}_{12}(\eta)$. As an effect, application of the ladder operator will change the sector of the quantum pendulum. Algebraically speaking this can be traced back to the fact that the intersection of the ideal J_{z_1,z_2} with $\mathcal{S}\otimes\mathcal{S}$ is not an ideal in $\mathcal{S}\otimes\mathcal{S}$ since $A^{(2)}$ is not central in $\mathcal{S}\otimes\mathcal{S}$.

6.1 Bethe ansatz groundstate

The Bethe ansatz ground state is a solution of the equation

$$\mathcal{M}_{21}(\eta)\Omega = 0. \tag{6.1}$$

Expanding $\mathcal{M}_{21}(\eta)=\eta^{\frac{1}{2}}(C^{(\frac{1}{2})}-\eta C^{(\frac{3}{2})})$ into powers we get

$$C^{(\frac{1}{2})} = (h_2^*U_1^{-1}+h_1^*U_2V_1^{-1}V_2)\,V_2^{-1} \tag{6.2}$$

$$C^{(\frac{3}{2})} = (h_2^*U_1Z_1+h_1^*U_2^{-1}Z_2^{-1}V_1^{-1}V_2)\,V_2^{-1}. \tag{6.3}$$

Hence, (**??**) is equivalent to the two equations

$$(h_2^*U_1^{-1}U_2^{-1}+h_1^*V_1^{-1}V_2)\tilde{\Omega}=0 \tag{6.4}$$

$$(U_1U_2Z_1-U_1^{-1}U_2^{-1}Z_2^{-1})\tilde{\Omega}=0 \tag{6.5}$$

where $\tilde{\Omega}=V_2^{-1}\Omega$. Setting $T=V_1^{-1}V_2$ we reformulate (**??**,**??**)

$$(h_2^*A^{(0)}+h_1^*T)\tilde{\Omega}=0 \tag{6.6}$$

$$(A^{(0)}-A^{(2)})\tilde{\Omega}=0 \tag{6.7}$$

and solve the first equation in a representation of the quotient algebra $\mathcal{C}$ modulo the ideal generated by the element $A^{(0)}-A^{(2)}$. Stated differently, we look at representations of the algebra generated by $\{A^{(2)},C_4,T\}$ and substitute in (**??**) $A^{(0)}$ by $A^{(2)}$. A faithful family of representations of that algebra labelled by two angles ϑ and δ is given by the following extensions of the Weyl-Schrödinger representations of $\mathcal{A}_\gamma$ on $\ell^2(\mathbb{Z})$:

$$\begin{aligned} C_4\cdot\phi(n) &= e^{i\vartheta}q^{-n}\phi(n) \\ T\cdot\phi(n) &= \phi(n-1) \\ A^{(2)}\cdot\phi(n) &= e^{i\delta}\phi(n). \end{aligned} \tag{6.8}$$

Theorem 8 *For all values of* ϑ *and* δ *there exists a solution* $\Omega_{\vartheta,\delta}$ *of (**??**) which is a complex-valued bounded function over* $\mathbb{Z}$*. It is a generalized eigenvector of* $\mathcal{M}_{11}(\eta)$ *whose eigenvalue is independent of* ϑ*, namely,*

$$\mathcal{M}_{11}(\eta)\Omega_{\vartheta,\delta} = e^{i\delta}\left(q^{-\frac{1}{2}}\eta^2 + (k^{-1}+k)\eta + q^{\frac{1}{2}}\right)\Omega_{\vartheta,\delta}. \tag{6.9}$$

In particular, the spectrum of the SG-integral has a band containing the interval $[-2(k+k^{-1}), 2(k+k^{-1})]$*.*

Proof: Since $h_2^* A^{(0)} + h_1^* T = (k^{-\frac{1}{2}} + k^{\frac{1}{2}} q^{\frac{1}{2}} C_4^*) A^{(0)} + (k^{-\frac{1}{2}} + k^{\frac{1}{2}} q^{\frac{1}{2}} C_4 A^{(0)} {A^{(2)}}^{-1}) T$ and upon substituting $A^{(0)}$ for $A^{(2)}$ equation (**??**) looks in the above representation, where $\tilde{\Omega}$ is a function over $\mathbb{Z}$,

$$(k^{-\frac{1}{2}} + k^{\frac{1}{2}} q^{n+\frac{1}{2}} e^{-i\vartheta})\, e^{i\delta}\, \tilde{\Omega}(n) + (k^{-\frac{1}{2}} + k^{\frac{1}{2}} q^{-n+\frac{1}{2}} e^{i\vartheta})\, \tilde{\Omega}(n-1) = 0.$$

Let us first consider the case $k=1$ and $\vartheta = \pi + (n+\frac{1}{2})\gamma$ for some $n \in \mathbb{Z}$. Then $\tilde{\Omega}(n) = \delta_n$ is a solution, where $\delta_n(m) = 1$ if $n = m$ and 0 otherwise. In the other case, $e^{i\vartheta} \neq -k^{-1} q^{n+\frac{1}{2}}$ for all $n \in \mathbb{Z}$, we obtain a recursion relation

$$\frac{\tilde{\Omega}(n-1)}{\tilde{\Omega}(n)} = -e^{i\delta} \frac{k^{-\frac{1}{2}} + k^{\frac{1}{2}} e^{-i\vartheta} q^{n+\frac{1}{2}}}{k^{-\frac{1}{2}} + k^{\frac{1}{2}} e^{i\vartheta} q^{-n+\frac{1}{2}}}. \tag{6.10}$$

In particular

$$\left|\frac{\tilde{\Omega}(n-m)}{\tilde{\Omega}(n)}\right|^2 = \frac{1 + k^2 + k(e^{i\vartheta} q^{n+m-\frac{1}{2}} + c.c.)}{1 + k^2 + k(e^{i\vartheta} q^{n-\frac{1}{2}} + c.c.)}$$

which shows that $\tilde{\Omega}(n)$ remains bounded for all n. Therefore a solution can always be constructed recursively. For the second case it is even unique up to a constant. It is not square summable, but this is to be expected. Now let $\tilde{\Omega}_{\vartheta,\delta}$ be a solution of the recursion relation for given ϑ and δ and set $\Omega_{\vartheta,\delta} = V_2 \tilde{\Omega}_{\vartheta,\delta}$. For that we should identify V_2 with a unitary operator in a Hilbert space which contains the representation space of representation (**??**). But this is in fact not neccessary since we can use the algebraic relations between V_2 and the elements of $\mathcal{C}$ to carry out all neccessary calculations. Let us first determine $A^{(1)}\Omega_{\vartheta,\delta}$.

$$\begin{aligned}
A^{(1)}\Omega_{\vartheta,\delta} &= (U_2 Z_2 U_1^{-1} + U_1 Z_1 U_2^{-1})\Omega_{\vartheta,\delta} + V_2 h_2 h_1^* V_1^{-1} V_2 \tilde{\Omega}_{\vartheta,\delta} \\
&\overset{??}{=} (U_2 Z_2 U_1^{-1} + U_1 Z_1 U_2^{-1})\Omega_{\vartheta,\delta} - V_2 h_2 h_2^* U_2^{-1} U_1^{-1} \tilde{\Omega}_{\vartheta,\delta} \\
&= \big(U_2 Z_2 U_1^{-1} + U_1 Z_1 U_2^{-1} - U_2 U_1^{-1} Z_2 - q^{-1} U_1^{-1} U_2^{-3} Z_2^{-1} \\
&\qquad - q^{-\frac{1}{2}}(k^{-1}+k) U_2^{-1} U_1^{-1}\big)\,\Omega_{\vartheta,\delta} \\
&\overset{??}{=} -q^{\frac{1}{2}}(k^{-1}+k) A^{(2)}\Omega_{\vartheta,\delta}
\end{aligned}$$

Since $A^{(2)}\Omega_{\vartheta,\delta} = q^{-\frac{1}{2}} V_2 A^{(2)} \tilde{\Omega}_{\vartheta,\delta} = e^{i\delta} q^{-\frac{1}{2}} \Omega_{\vartheta,\delta}$ and, similarly, $A^{(0)}\Omega_{\vartheta,\delta} = q A^{(2)}\Omega_{\vartheta,\delta}$ one obtains from (**??**) the result for $\mathcal{M}_{11}(\eta)$. q.e.d.

Note that, in case $k=1$ and $\vartheta = \pi + (n+\frac{1}{2})\gamma$ the support of $\tilde{\Omega}$ is 1.

6.2 Bethe ansatz equations

The following theorem is the specification to the case of shortest periodicity of the general result derived in [**?**] for the discrete sine Gordon model.

Theorem 9 *Let $\Omega_{\vartheta,\delta}$ be a Bethe ansatz-groundstate. Then*

$$\Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1}) = \mathcal{M}_{12}(\eta_1)\ldots\mathcal{M}_{12}(\eta_{n-1})\Omega_{\vartheta,\delta}$$

is a (generalized) eigenvector of $\mathcal{F}(\eta)$ provided the spectral parameters $\eta_1,\ldots,\eta_l$ are pairwise different and satisfy the so-called Bethe ansatz-equations

$$\prod_{j\neq i=1}^{n-1} \frac{\eta_j q - \eta_i}{\eta_i q - \eta_j} = \frac{a_\delta(\eta_j)}{a_\delta^*(\eta_j)}, \tag{6.11}$$

where

$$a_\delta(\eta) = e^{i\delta}(q^{-\frac{1}{2}}\eta^2 + (k+k^{-1})\eta + q^{\frac{1}{2}}) \tag{6.12}$$

*is the eigenvalue of $\mathcal{M}_{11}(\eta)$ on $\Omega_{\vartheta,\delta}$, cf. (**??**). The corresponding eigenvalue $\mathcal{E}_\delta(\eta,\eta_1,\ldots,\eta_{n-1})$ of $\mathcal{F}(\eta)$ is given by $\mathcal{E}_\delta = f_\delta + f_\delta^*$ where*

$$f_\delta(\eta,\eta_1,\ldots,\eta_{n-1}) = a_\delta(\eta)\prod_{j=1}^{n-1}\frac{\eta_j q^{\frac{1}{2}} - \eta q^{-\frac{1}{2}}}{\eta_j - \eta}. \tag{6.13}$$

The eigenvalue $E_\delta(\eta_1,\ldots,\eta_{n-1})$ of the SG-integral is given by the linear coefficient of the expansion of $\mathcal{E}_\delta(\eta,\eta_1,\ldots,\eta_{n-1})$ in η, that is $E_\delta = -\partial_\eta f_\delta|_{\eta=0} + c.c.$ (the complex conjugation not effecting the η_i). It is easily seen that

$$\partial_\eta f_\delta|_{\eta=0} = e^{i\delta} q^{\frac{n-1}{2}} \left(k + k^{-1} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{j=1}^{n-1} \eta_j^{-1} \right). \tag{6.14}$$

Since we can vary δ, all these become bands of the SG-integral around 0.

6.3 Bethe Ansatz for reductions of the SG-integral

If we want to specialize the above to reductions of the SG-integral we find that the special form of the ladder operator imposes restrictions on the choice of the reduction parameters. In particular, it will turn out that we get the same constraint on b as in Corollary **??**.

First note that, the ladder operator has the form $\mathcal{M}_{12}(\eta) = \mathcal{B}(\eta)U_2^{-1}V_2$ where $\mathcal{B}(\eta) \in \mathcal{C}$. Therefore,

$$A^{(2)}\mathcal{M}_{12}(\eta) = q^{-\frac{1}{2}}\mathcal{M}_{12}(\eta)A^{(2)}, \tag{6.15}$$

$$A^{(0)}\mathcal{M}_{12}(\eta) = q^{\frac{1}{2}}\mathcal{M}_{12}(\eta)A^{(0)}. \tag{6.16}$$

Corollary 3 *Let $\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1})$ be as in Theorem* **??**. *Then,*

$$H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k)\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1}) = E_\delta(\eta_1,\dots,\eta_{n-1})\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1}) \qquad (6.17)$$

where

$$E_\delta(\eta_1,\dots,\eta_{n-1}) = -2(k+k^{-1})\cos(\frac{n-1}{2}\gamma+\delta) \qquad (6.18)$$
$$+4\sin(\frac{n-1}{2}\gamma+\delta)\,\sin\frac{\gamma}{2}\sum_{j=1}^{n-1}\eta_j^{-1}(\gamma).$$

provided the η_j satisfy the Bethe ansatz-equations

$$\prod_{j\neq i=1}^{n-1}\frac{\eta_j q-\eta_i}{\eta_i q-\eta_j} = e^{2i\delta}\frac{q^{-\frac{1}{2}}\eta_j^2+\eta_j(k+k^{-1})+q^{\frac{1}{2}}}{q^{\frac{1}{2}}\eta_j^2+\eta_j(k+k^{-1})+q^{-\frac{1}{2}}}. \qquad (6.19)$$

Proof: To obtain an eigenvalue Φ for the reduction of the SG-integral which is defined by z_1, z_2 we must before all make sure that $w\Phi = z_1\Phi$ and $A^{(2)}\Phi = z_2\Phi$. Due to the relations (**??**,**??**) $A^{(2)}\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1}) = z_2\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1})$ implies,

$$A^{(2)}\tilde{\Omega}_{\vartheta,\delta} = z_2 q^{\frac{n}{2}}\tilde{\Omega}_{\vartheta,\delta} \qquad (6.20)$$
$$w\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1}) = q^{\frac{n}{2}}\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1}) \qquad (6.21)$$

Consequently, we must choose $e^{i\delta} = q^{\frac{n}{2}}z_2$ and $z_1 = q^{\frac{n}{2}}$. Hence, for $b = z_1 + c.c. = 2\cos\frac{n}{2}\gamma$ and $a = z_1z_2 + c.c. = 2\cos\delta$ we may apply the last theorem to obtain generalized eigenvectors and (**??**) to obtain their eigenvalues for $H(a,b,k)$ as stated. q.e.d.

We call the (generalized) eigenvalue (**??**) and the corresponding (generalized) eigenvector $\Phi_{\vartheta,\delta}(\eta_1,\dots,\eta_{n-1})$ of $H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k)$ simply Bethe ansatz-eigenvalue and Bethe ansatz-eigenvector, respectively.

The corollary shows a peculiarity of the reductions. Due to the relations (**??**,**??**) the Bethe Ansatz does not furnish us with infinitely many eigenfunctions and -values. In particular, in order to obtain the Bethe ansatz-eigenvectors and -values for $H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k)$ we have to solve for a Bethe ansatz-groundstate which is, for $n > 1$ not an eigenstate of $H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k)$.

6.3.1 *Solution for $n = 1$*

The solution for $n = 1$ is contained in the corollary without explicit mentioning. The Bethe ansatz-eigenvector is the Bethe ansatz-groundstate, so that there are no Bethe ansatz-equations to solve, and the corresponding eigenvalue is given by expression (**??**) without the sum, i.e. by

$$E_\delta = -2(k+k^{-1})\cos\delta.$$

This solution, specialized to the value for the QP-integral which is $\delta = \frac{\gamma}{2}$, is shown in Figure **??**; it is the same solution as in [**?**] where it was depictured for rational values of $\frac{\gamma}{2\pi}$.

6.3.2 *Solution for* $n = 2$

For $n = 2$ there is a single Bethe ansatz-equation, namely

$$1 = e^{2i\delta} \frac{q^{-\frac{1}{2}}\eta_1^2 + \eta_1(k + k^{-1}) + q^{\frac{1}{2}}}{q^{\frac{1}{2}}\eta_1^2 + \eta_1(k + k^{-1}) + q^{-\frac{1}{2}}}. \tag{6.22}$$

We solve this equation for $\delta = \gamma$ which corresponds to the QP-integral. In this case it may be written with $\eta = \eta_1$ as

$$\eta^2(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) + \eta(k + k^{-1})(q - q^{-1}) + q^{\frac{3}{2}} - q^{-\frac{3}{2}} = 0.$$

It is merely of second degree in η and has two real solutions:

$$\eta_\pm = -(k + k^{-1}) \cos\frac{\gamma}{2} \pm \sqrt{((k + k^{-1})^2 - 4)\cos^2\frac{\gamma}{2} + 1}.$$

It follows that

$$E_\gamma(\eta_\pm) = -2(k+k^{-1}) \cos\frac{\gamma}{2}(1-2\sin^2\frac{\gamma}{2}) \mp 4\sin^2\frac{\gamma}{2}\sqrt{((k + k^{-1})^2 - 4)\cos^2\frac{\gamma}{2} + 1}. \tag{6.23}$$

These two solutions are shown in Figure **??**.

6.4 Positioning of the Bethe ansatz-eigenvalues in bands of the spectrum of the QP-integral

Like in Section 4 one may ask, where in the bands, which exist for rational flux, the eigenvalues obtained by the algebraic Bethe ansatz, i.e. by Corollary **??**, may be found. We shall verify now that for $k > 1$ too, the Bethe ansatz-eigenvalues of the QP-integral lie at h-band edges, provided the flux is rational. Hence, we assume for the rest of this section that $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M, N) = 1$.

Recall that the first step to determine the positioning of a Bethe ansatz-eigenvalue of $H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k)$ is to determine the Bloch-parameters at which they exist, that is, to determine the eigenvalues of P_1^N and P_2^N on the corresponding Bethe ansatz-eigenvector $\Phi_{\vartheta,\delta}(\eta_1, \dots, \eta_{n-1})$.

Clearly, $e^{iN\vartheta}$ is the eigenvalue of C_4^N on $\tilde\Omega_{\vartheta,\delta}$. For the above value of $\frac{\gamma}{2\pi}$ and $k > 1$, the vector $\tilde\Omega_{\vartheta,\delta}$ becomes N-periodic. Hence it is an eigenvector of T^N. We denote the corresponding eigenvalue by β. Clearly, $\beta = \frac{\tilde\Omega_{\vartheta,\delta}(n-N)}{\tilde\Omega_{\vartheta,\delta}(n)}$, independently of n, so that we obtain from (**??**) an equation among the eigenvalues of $C_4^N, T^N, A^{(2)}$ (these operators are all central in $\mathcal{C}$), namely

$$\beta = (-e^{i\delta})^N \prod_{n=0}^{N-1} \frac{k^{-1} + e^{-i\vartheta}q^{\frac{1}{2}-n}}{k^{-1} + e^{i\vartheta}q^{n+\frac{1}{2}}}.$$

Using that

$$\prod_{m=0}^{N-1}(a+q^{m+\frac{1}{2}}) = a^N - (-1)^{M+N},$$

which may be seen by comparison of the roots, we obtain that, for $k>1$,

$$\beta = (-1)^N e^{iN\delta} e^{-iN\vartheta} \frac{(-1)^N e^{iN\vartheta} - (-1)^M k^N}{(-1)^N - (-1)^M e^{iN\vartheta} k^N}. \tag{6.24}$$

Proposition 4 *Let* $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M,N)=1$, *and* $\Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1})$ *be a Bethe ansatz-eigenvector of* $H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k)$. *Then*

$$P_1^N \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1}) = (-1)^{Mn} e^{iN\vartheta} \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1})$$

and, for $k>1$,

$$P_2^N \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1}) = e^{i\tilde{\theta}_2} \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1})$$

with

$$e^{i\tilde{\theta}_2} = -e^{iN\delta} \frac{(-1)^{Mn} e^{i\tilde{\theta}_1} - (-1)^{M+N} k^N}{1 - (-1)^{M(n-1)+N} e^{i\tilde{\theta}_1} k^N} \tag{6.25}$$

and $e^{i\tilde{\theta}_1} = (-1)^{Mn} e^{iN\vartheta}$.

Proof: Since $\Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1})$ is of the form $\mathcal{B}'(\eta_1,\ldots,\eta_{n-1}) U_2^{1-n} V_2^n \tilde{\Omega}_{\vartheta,\delta}$, with $\mathcal{B}'(\eta_1,\ldots,\eta_{n-1}) \in \mathcal{C}$, the eigenvalues of P_1^N and P_2^N can be obtained from the eigenvalues of C_4^N, T^N, $A^{(2)}$, and w upon using their relations with $U_2^{1-n}V_2^n$. These relations are

$$wU_2^{1-n}V_2^n = q^{\frac{n}{2}} U_2^{1-n} V_2^n w, \quad C_4 U_2^{1-n} V_2^n = q^{-n} U_2^{1-n} V_2^n C_4,$$

$$A^{(2)} U_2^{1-n} V_2^n = q^{-\frac{n}{2}} U_2^{1-n} V_2^n A^{(2)}, \quad T U_2^{1-n} V_2^n = q^{-\frac{n-1}{2}} U_2^{1-n} V_2^n T.$$

Hence, for $P_1 = wC_4$ one obtains

$$P_1^N \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1}) = (-1)^{Mn} e^{iN\vartheta} \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1})$$

which shows the first statement. For $P_2 = q^{\frac{1}{2}} P_1 T$ we obtain

$$P_2^N \Phi_{\vartheta,\delta}(\eta_1,\ldots,\eta_{n-1}) = \mathcal{B}'(\eta_1,\ldots,\eta_{n-1}) U_2^{n-1} V_2^n (P_1 T)^N \tilde{\Omega}_{\vartheta,\delta}.$$

Since $(P_1T)^N = q^{N\frac{N-1}{2}} P_1^N T^N$ and $q^{N\frac{N}{2}} = (-1)^{MN} = (-1)^{M+N+1}$ (as M and N are coprime) the eigenvalue of P_2^N is given by

$$e^{i\tilde{\theta}_2} = (-1)^{N+1} e^{iN\vartheta} \beta,$$

provided that $k>1$. We can combine this with (**??**) obtaining (**??**). q.e.d.

The following theorem shows that the Blochparameters just determined furnish in case of the QP-integral $H_{QP}(n,k) = H(2\cos\frac{n}{2}\gamma, 2\cos\frac{n}{2}\gamma, k)$ a critical set of the off-set function, for $k>1$ too.

Theorem 10 *Let $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M,N) = 1$, and $k > 1$. The off-set function $h(2\cos\frac{n\gamma}{2}, 2\cos\frac{n\gamma}{2}, k)$ of the QP-integral takes for odd M its maximum and for even M its minimum on the one-dimensional subset of the torus*

$$\{\vec{\theta} \in S^1 \times S^1 | e^{iN\theta_2} = -(-1)^{Mn} \frac{(-1)^{Mn} e^{iN\theta_1} - (-1)^{M+N} k^N}{1 - (-1)^{M(n-1)+N} e^{iN\theta_1} k^N}\}. \tag{6.26}$$

In particular, the Bethe ansatz-eigenvalues of the QP-integral are h-band edges.

Proof: Inserting $r_a = r_b = 1$ and $\theta_a = \theta_b = \frac{n\gamma}{2}$ into (**??**,**??**) one obtains that the partial derivatives of the off-set function vanish on the above set. Further explicit calculation shows that $h(2\cos\frac{n}{2}\gamma, 2\cos\frac{n}{2}\gamma, k)$ takes on this set the value $(-1)^{M-1}2(k^N + k^{-N})$ and that this value is an absolute extremum. The set (**??**) consisting of those $\vec{\theta}$ for which $N\vec{\theta}$ satisfy (**??**) it follwos that the Bethe ansatz-eigenvalues of the QP-integral $H(2\cos\frac{n}{2}\gamma, 2\cos\frac{n}{2}\gamma, k)$ lie at h-band edges.q.e.d.

This can be used to generalize Theorem **??** to $k > 1$.

Theorem 11 *For $\frac{\gamma}{2\pi} = \frac{M}{2n}$, $(M, 2n) = 1$, the Bethe ansatz-eigenvalues of $H_{QP}(n,k)$ are points where h-bands touch. Moreover, they are interior points in the spectrum of the anisotropic Hofstadter Hamiltonian $H_{Hof}(k^2, 2\gamma)$.*

Proof: The last theorem shows that the eigenvalues of $H_{QP}(n,k)$ obtained in Corollary **??** are h-band edges, if $k > 1$. We assert first, that for $k = 1$ they are h-band edges, too. Proposition **??** shows that if $k = 1$ and $\vartheta = \pi + (n' + \frac{1}{2})\gamma$, $n' \in \mathbb{Z}$, then the eigenvalue of P_1^N is equal to $(-1)^{M(n+1)}$ (here $N = 2n$). In particular, P_1^N has in both representations, that of Corollary **??** and that of Corollary **??**, the same eigenvalue. This eigenvalue being the first Blochparameter and the second Blochparameter being unimportant (by Lemma **??**) it follows that all eigenvalues of $H_{QP}(n,1)$ obtained by Corollary **??** are among the eigenvalues obtained by Corollary **??**. But of those we know already that they lie at h-band edges. Moreover, by Theorem **??** the eigenvalues are for $k = 1$ band touching points. Recall that at $\frac{\gamma}{2\pi} = \frac{M}{2n}$, $H_{QP}(n,k) = kH_{Hof}(k^2, 2\gamma)$. Using the wellknown fact that no gaps of $H_{Hof}(k^2, 2\gamma)$ open up or close upon varying k, see e.g. [**?**] for a proof, we conclude that the eigenvalues in question remain h-band touching points for $k > 1$, too, because otherwise a gap would appear. Since h-band touching points lie in the interior of the spectrum the last statement is clear. q.e.d.

Not only the last theorem suggests that the Bethe ansatz-eigenvalues of the QP-integral play a special role in the spectrum. Investigating Figures 2 and 3 one sees that the flux-dependent curves which describe the Bethe ansatz-eigenvalues almost never seem to intersect the bands at their edges. This leads us to conjecture:

- If $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M,N) = 1$, and $N \geq 2n$ then the Bethe ansatz-eigenvalues of the QP-integral are points in the spectrum where h-bands touch.

Looking at Figure 3 one sees that for $\frac{\gamma}{2\pi} = \frac{M}{4}$, $M = 1, 3$, an extra h-band touching is present at $E = 0$. This is easily explained by the observation that $\frac{\gamma}{2\pi} = \frac{M}{4}$ is value of the flux at which the QP-integral coincides up to a multiplicative constant with the Hofstadter Hamiltonian. The h-band touching there is thus the same as the one of the Hofstadter Hamiltonian at $E = 0$. As already mentionned the latter gives rise to a conical singularity in the graph of the band function. These conical singularities do not occurr at Bethe ansatz-eigenvalues, since there the relevant extremum of the off set function forms a one-dimensional set. The h-band touching occuring at Bethe ansatz-eigenvalues is therefore topologically different from the h-band touching occurring at $\frac{\gamma}{2\pi} = \frac{M}{4}$, $M = 1, 3$, $E = 0$.

7 Comparison

We have presented two methods to obtain eigenvalues and -functions for a three parameter family of operators $H(a, b, k)$. The first one was to look for polynomial eigenfunctions, the second the algebraic Bethe ansatz. Both approaches have limited applicability. But both apply to $H(a, b, k)$ with $k = 1$ and $b = 2\cos\frac{n\gamma}{2}$, $n \in \mathbb{N}$, and $a \in [-2, 2]$. In this situation both approaches give rise to families of eigenvalues and eigenvectors which are at least continuous in the flux, thus, in particular hold for irrational values of the flux as well. The advantage of the polynomial Bethe ansatz lies in its simplicity. It reduces the eigenvalue problem to one of finite matrices. But the algebraic Bethe ansatz is more powerful, since it applies to the case $k \neq 1$ as well.

Comparing the two approaches the first remarkable fact is that the restriction on the parameter b, which was neccessary for the existence of solutions which are at least continuous in the flux and hold for irrational values as well, is in both approaches identical. Second, restricting the Bethe ansatz-eigenvector obtained by the algebraic Bethe ansatz to the case $k = 1$ one finds that the Bethe ansatz-groundstate Ω has, in the appropriate representation, support of length 1. Moreover, the ladder operator $\mathcal{M}_{12}(\eta)$ enlarges the support by one so that the eigenstates $\Phi_{n-1}(\eta_1, \ldots, \eta_{n-1})$ have support of length smaller or equal to n, or equivalently, their Fouriertransforms are Laurent polynomials, in fact, polynomials, of degree $n - 1$. In other words, the algebraic Bethe ansatz furnishes for $k = 1$ also polynomial eigenfunctions. Of course, these eigenfunctions have to be among those obtained by solving the eigenvalue equation for the $n \times n$ matrix $\tilde{H}$ of (**??**). So the question arrises, how are the two kind of Bethe ansatz-equations related. They look rather different. This difference steems from the fact that the solutions of the functional Bethe ansatz-equations furnish eigenfunctions whereas those of the algebraic Bethe ansatz yield admissible products of ladder operators. Hence they are related as follows: If $\eta_1, \ldots, \eta_{n-1}$ solve the algebraic Bethe ansatz-equations then the zeros of the Fouriertransform of

$$\mathcal{M}_{12}(\eta_1) \ldots \mathcal{M}_{12}(\eta_{n-1})\Omega_{\delta,\vartheta}$$

solve the functional Bethe ansatz-equations.

Bibliography

[AvMS90] J. Avron, P. H. M. v. Mouche, and B. Simon. On the Measure of the Spectrum for the Almost Mathieu Operator. *Comm. Math. Phys.*, 132:103 – 118, 1990.

[BBR96] V. Bazhanov, A. Bobenko, and N. Reshetikhin. Quantum discrete sine-gordon model at roots of 1: Integrable quantum system on the integrable classical background. *Comm. Math. Phys.*, 175:377–400, 1996.

[Bel94] J. Bellissard. Lipshitz Continuity of Gap Boundaries for Hofstadter-like Spectra. *Comm. Math. Phys.*, 160:599 – 613, 1994.

[Bet31] H. Bethe. Zur Theorie der Metalle; I. Eigenwerte und Eigenfunktionen der Linearen Atomkette. *Z. d. Phys.*, 71:205–226, 1931.

[BF95] V. Buslaev and A. Fedotov. Complex WKB method for Harper's equation I. *St. Petersburg Math. Journal*, 6(3):59–63, 1995.

[BKP93] A. Bobenko, N. Kutz, and U. Pinkall. The Discrete Quantum Pendulum. *Physical Letters A*, 177:399–404, 1993.

[BP96] A. Bobenko and U. Pinkall. Discrete Surfaces with constant Negative Gaussian Curvature and the Hirota Equation. *J. Diff. Geom.*, 43:527–611, 1996.

[BPar] A. Bobenko and U. Pinkall. Discretization of surfaces and integrable systems. In *Discrete Integrable Geometry and Physics*. Oxford University Press, to appear.

[BS82] J. Bellissard and B. Simon. Cantor Spectrum for the Almost Mathieu Equation. *J. Func. An.*, 48:408 – 419, 1982.

[CEY90] M.-D. Choi, G. A. Elliott, and N. Yiu. Gauss polynomials and the rotation algebra. *Inv. Math.*, 99:225 – 246, 1990.

[CN96] H.W. Capel and F. W. Nijhoff. Integrable quantum mappings. In L. Vinet D. Levi and P. Winternitz, editors, *Symmetries and Integrability of Difference Equations*. CRM Proceedings and Lecture Notes, vol. **9l**,pp. 37–49, (AMS, Providence 1996), 1996.

[EK95] C. Emmrich and N. Kutz. Doubly discrete Lagrangian systems related to the Hirota and the sine-Gordon equation. *Phys. Lett.*, A 201:156–160, 1995.

[Ell82] G. Elliot. Gaps in the spectrum of an almost periodic Schrödinger operator. *C. R. Math. Rep. Acad. Sci. Canada*, 4:255–259, 1982.

[ER93] G. A. Elliot and M. Rordam. The Automorphism group of the irrational rotation C^*-algebra. *Comm. Math. Phys.*, 155:3–26, 1993.

[Fad93] L. D. Faddeev. The Bethe ansatz. Sfb 288 Preprint Nr. 70, available at http://www-sfb288.math.tu-berlin.de, 1993.

[FK95] L. D. Faddeev and R. M. Kashaev. Generalized Bethe Ansatz Equations for the Hofstadter Problem. *Comm. Math. Phys.*, 169:181–191, 1995.

[FVar] L. D. Faddeev and A. Yu. Volkov. Algebraic quantization of integrable models in discrete space-time. In *Discrete Integrable Geometry and Physics*. Oxford University Press, to appear.

[Hir77] R. Hirota. *Nonlinear Partial Difference Equations III* : Discrete sine-gordon equation. *J.Phys.Soc.Japan*, 43(6):2079–2086, Dec. 1977.

[HKW94] Y. Hatsugai, M. Kohmoto, and Y.S. Wu. Explicit Solutions of the Bethe Ansatz Equations for the Bloch electrons in a Magnetic Field. *Phys. Rev. Lett.*, 73, 1994.

[Hof76] D. Hofstadter. Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. *Phys. Rev. B*, 14, 1976.

[HS89] B. Helffer and S. Sjöstrand. Semi-classical analysis for Harper's equation III, Cantor structure of the spectrum. *Mém. Soc. Math. France (N.S.)*, 117(39):1 – 124, 1989.

[Jit95] Svetlana Ya. Jitomirskaya. Almost Everything about the Almost Mathieu Operator, II. In Daniel Iagolnitzer, editor, *XIth International Congress of Mathematical Physics*, pages 373 – 382, Boston, 1995. International Press.

[Kat80] T. Kato. *Perturbation Theory for Linear Operators*. Springer, 1980.

[KS96] C. Kreft and R. Seiler. Models of the Hofstadter type. *J. Math. Phys.*, 37:5207 – 5243, 1996.

[Kut94] N. Kutz. On the spectrum of the quantum pendulum . *Phys. Lett. A*, 187:365–372, 1994.

[Kutar] N. Kutz. Lagrangian description of doubly discrete sine-Gordon type models . In *Discrete Integrable Geometry and Physics*. Oxford University Press, to appear.

[Las94] Y. Last. Zero Measure Spectrum for the Almost Mathieu Operator. *Comm. Math. Phys.*, 164:421 – 432, 1994.

[Las95] Y. Last. Almost Everything about the Almost Mathieu Operator, I. In Daniel Iagolnitzer, editor, *XIth International Congress of Mathematical Physics*, pages 366 – 372, Boston, 1995. International Press.

[MV91] J. Moser and A. P. Veselov. Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Comm. Math. Phys.*, 139:217–243, 1991.

[Poh76] K. Pohlmeyer. Integrable Hamiltonian Systems and Interactions through quadratic constraints. *Comm. Math. Phys.*, 46:207–221, 1976.

[RSar] T. Richter and R. Seiler. Charge Transport in the Discretized Landau Model . In *Discrete Integrable Geometry and Physics*. Oxford University Press, to appear.

[Sak85] J. Sakurai. *Modern Quantum Mechanics*. Benjamin/Cummings Publishing Company, 1985.

[Shu94] M.A. Shubin. Discrete magnetic Laplacian. *Comm. Math. Phys.*, 164:259–275, 1994.

[Sin87] Ya. G. Sinai. Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential. *J. Stat. Phys.*, 46:861–909, 1987.

[Skl83] E. K. Sklyanin. *Zap. Nauch. Semin. LOMI*, 134:112, 1983.

[Sur89] Yu. B. Suris. Integrable Mappings of the Standard Type. *Funct. Anal. App.*, 23:74–79, 1989.

[TF79] L.A. Takhtadzhan and L.D. Faddeev. The Quantum Method of the Inverse Problem and the Heisenberg XXZ Model. *Russ. Math. Surveys*, 34(5):11–68, 1979.

[Tho83] D. J. Thouless. Bandwidth for a quasiperiodic tight-binding model. *Phys. Rev. B*, 28:4272–4276, 1983.

[VF92] A.Yu. Volkov and L. D. Faddeev. Quantum Inverse Scattering Method. *Theor. Math. Phys.*, 92(2):207–214, August 1992.

[Vol92] A.Yu. Volkov. Quantum Volterra Model. *Phys. Lett. A*, 167:345–355, 1992.

[Wil87] Michael Wilkinson. An exact renormalization group for Bloch electrons in a magnetic field. *J. Phys. A*, 20:4337 – 4354, 1987.

[WO93] Wegge-Olson. *K-theory of C^*-algebras.* Oxford University Press, 1993.

[WZ94] P. B. Wiegmann and A. V. Zabrodin. Quantum group and magnetic translations, Bethe ansatz for the Asbel-Hofstadter problem. *Nucl. Phys. B*, 422:495–514, 1994.

[WZ95] P. B. Wiegmann and A.V. Zabrodin. Algebraization of difference eigenvalue equations related to $U_q(sl_2)$. *Nucl. Phys. B*, 451:699–724, 1995.

[Zab95] A.V. Zabrodin. Cyclic Representations of $U_q(sl_2)$ in Hofstadter model. In T. Lulek, W. Florek, and S. Walcerz, editors, *Symmetry and Structural Properties of Condensed Matter.* World Scientific, 1995.

[Zak64] J. Zak. Magnetic translation group. *Phys. Rev. A*, 134:1602–1611, 1964.

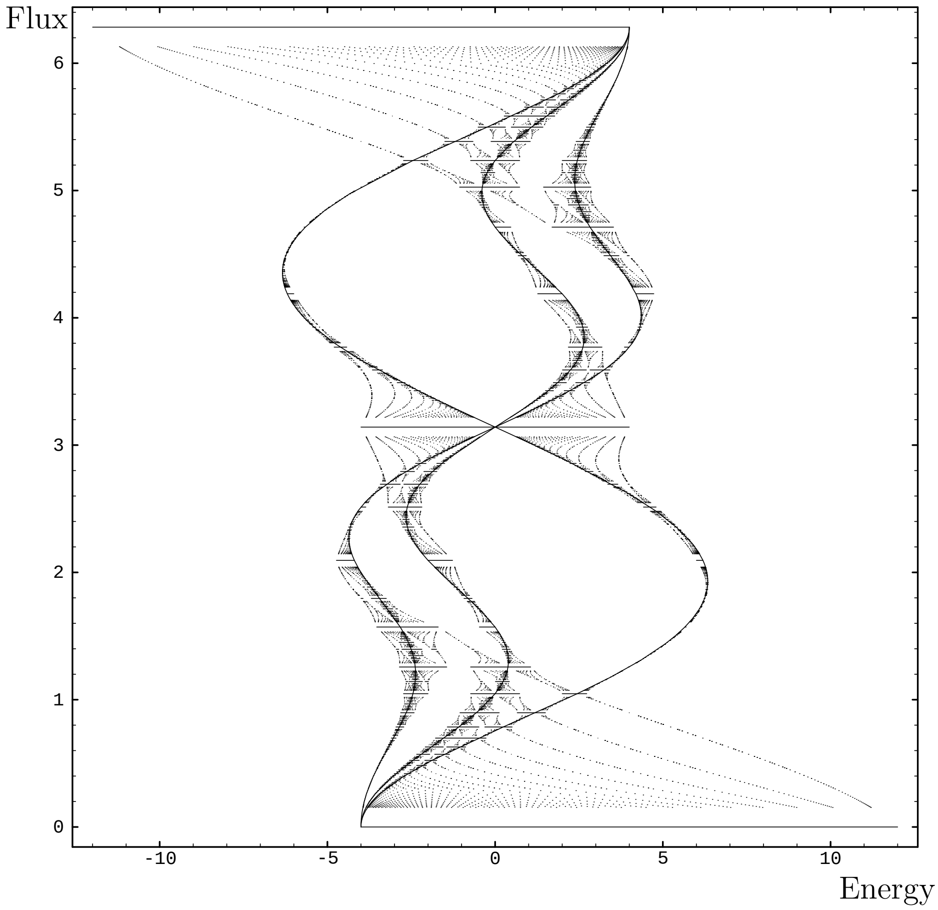


FIG. 1. Flux-dependence of the spectrum of $H_{QP}(3,1)$ for rational $\frac{\gamma}{2\pi}$ and Bethe ansatz-eigenvalues

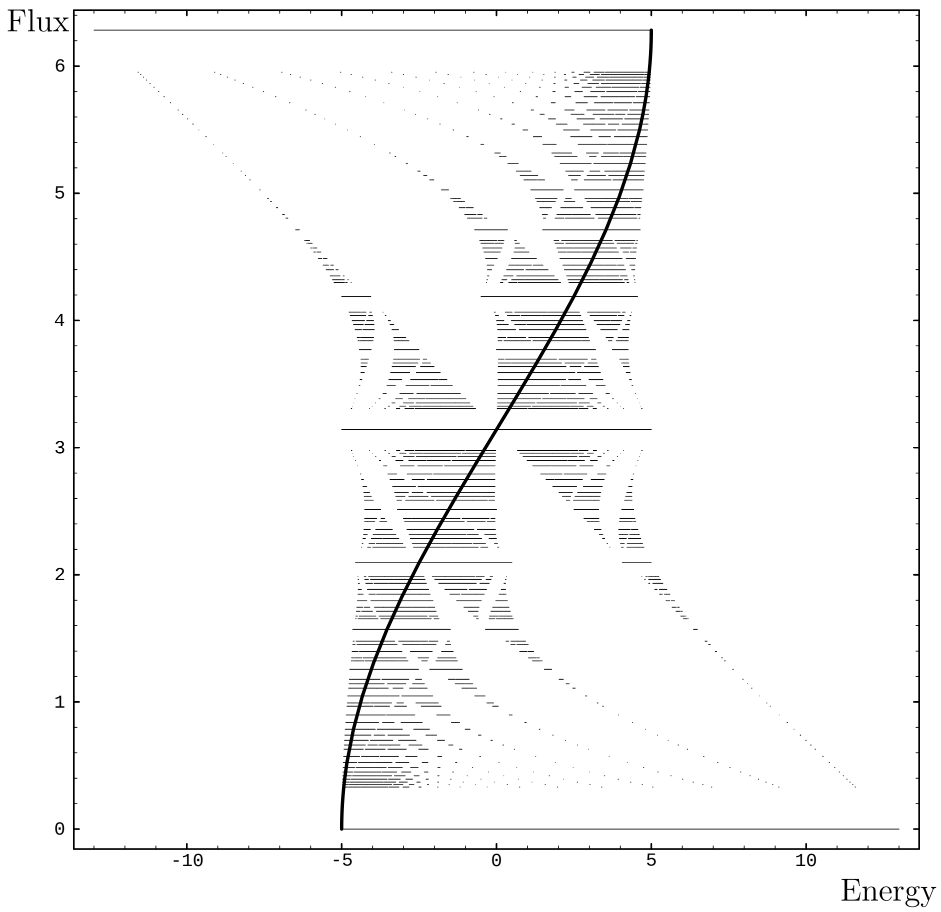


FIG. 2. Flux dependence of the spectrum of $H_{QP}(1,2)$ for rational $\frac{\gamma}{2\pi}$ and Bethe ansatz-eigenvalues

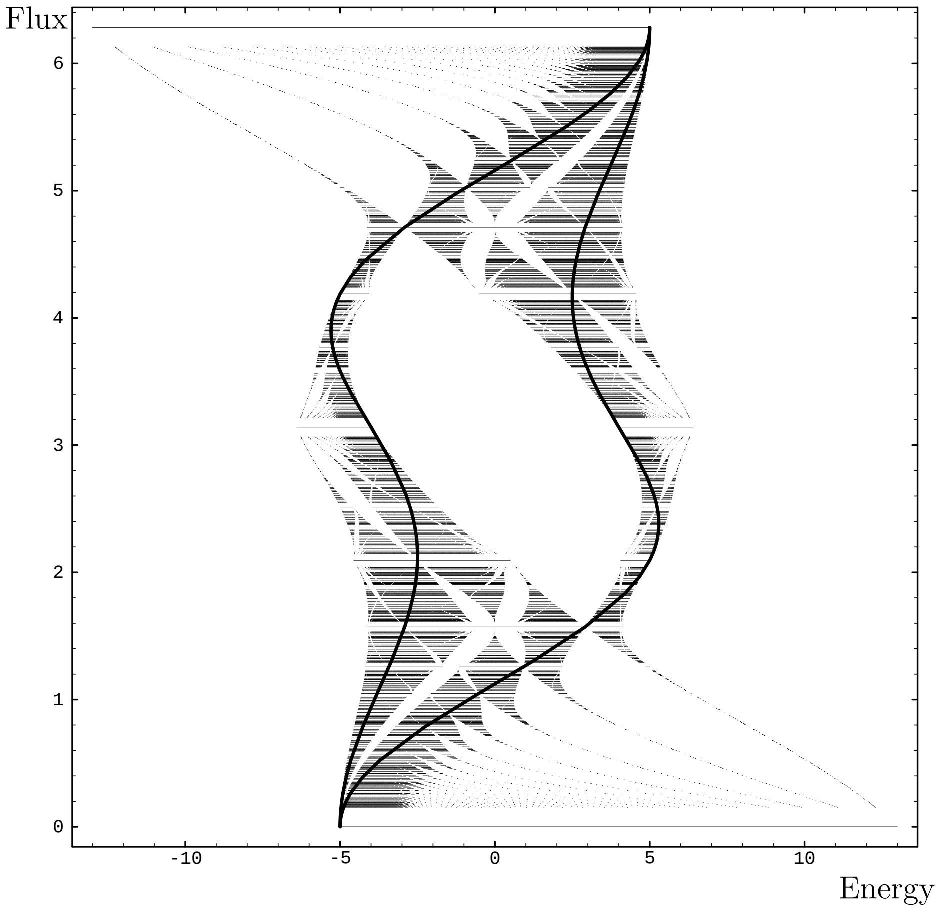


FIG. 3. Flux dependence of the spectrum of $H_{QP}(2,2)$ for rational $\frac{\gamma}{2\pi}$ and Bethe ansatz-eigenvalues