1

Lagrangian description of doubly discrete sine-Gordon type models

Nadja Kutz

1 Introduction of the model

The following work will give some new additions to results achieved in a joint work by Claudio Emmrich and the author [?]. A stress will be put on a different motivation of the in [?] derived symplectic structure, by translating the idea of a symplectic current as presented in [?] for solutions to classical equations of the type:

$$\Box\phi = V'(\phi), \tag{1.1}$$

with $\Box$ being the standard laplacian on spacetime and ϕ being a scalar field, into a discrete analogue.

We will investigate dynamics, which are associated to a lattice analogue of $1+1$-dimensional Minkowski spacetime (see e.g.[?], [?],[?],[?], [?], [?]). Discrete Minkowski spacetime will be realized by a $\mathbb{Z}^2$-lattice L, where the edges point into the light cone directions, hence L will also be called "light cone lattice".

Discrete or lattice dynamics will be defined by an evolution equation of the following type:

$$g_u - g_d - V'(g_l - g_r) = 0, \tag{1.2}$$

where u, d, l and r denote up, down, left and right, respectively and $V'(x)$ is the derivative of $V : \mathbb{R} \to \mathbb{R}$. The fields $g : L \to \mathbb{R}$ are associated to the vertices of the lattice. If we start with initial data on a Cauchy path $\mathcal{C}$ (see fig. 1) the local evolution given by **??** will determine the function g on the whole lattice.

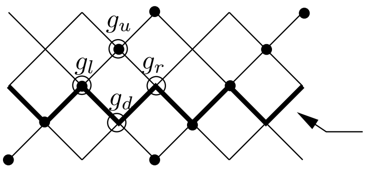

Figure 1

Assume $V(-x) = V(x)$. Consider the following redefinition of the fields g along every second diagonal in the lightcone lattice L:

$$\tilde{g}(\bullet) := -g(\bullet) \qquad \tilde{g}(\times) := g(\times) \tag{1.3}$$

where the symbols $\bullet$ and $\times$ are placeholders for the respective vertices as given in fig. 1. After such a redefinition the above equation reads

$$\tilde{g}_u - \tilde{g}_l - \tilde{g}_r + \tilde{g}_d = V'(\tilde{g}_l + \tilde{g}_r) - (\tilde{g}_l + \tilde{g}_r). \tag{1.4}$$

The LHS is a discrete analog of the standard laplacian. If

$$V'(x) = -i \ln\left(\frac{1 + ke^{ix}}{k + e^{ix}}\right)$$

then equation (**??**) is the wellkown integrable Hirota equation [**?**]. Without the above redefinition but with the same potential the corresponding systems are related to discrete analogues of the mKdV [**?**] and Volterra model [**?**]. In the forthcoming we will mainly studying system (**??**). With appropriate boundary conditions most of the discussed features can be taken over to the redefined system (**??**).

The difference of two spacelike adjacent values of g, i.e. $p := g_l - g_r$ for the potential above describes (modulo the above redefinition along the diagonals) the angle between the tangent directions of a discrete K-surface in the Tchebycheff parametrization [**?**],[**?**]. Equation **??** (**??** resp.) as well as the equation which one obtains for the above difference variables possess quantum analogues ([**?**],[**?**]). The equation of motion for the "quantized" angles of a K-surface proves to be a beautiful example in the theory of quantum groups [**?**, **?**]. Moreover a reduction of it gives an important model in solid state physics (see e.g. [**?**],[**?**]).

2 Discrete Forms on space time

Definition 2.1 *A map $f : \mathcal{V} \to \mathcal{V}$ from the set of vertices of L into some vector space V is a* **0-form on L with values in V**.

Definition 2.2 *A map $h : \mathcal{E} \to \mathcal{V}$ from the set of oriented edges of L into some vector space V with $h(-e) = -h(e)$, $e \in \mathcal{E}$ is a* **1-form on L with values in V**.

The above definitions are straightforward extensions of the usual definitions of k-forms on a manifold, i.e. we interprete vertices as points and oriented edges as discrete tangent vectors. The light cone lattice is a very simple space time graph as being imbedded in $\mathbb{R}^2$. Although it is possible to extend the below notions to more arbitrary graphs, we will use in the following the specific properties of L in order to avoid lengthy definitions. Considering the simple structure of $\mathbb{R}^2$, it is clear what is meant by adding neighbouring edges with the right orientation in order to form a chain or cycle. A cycle around a face of the lightcone lattice L shall be called **fundamental cycle.** Integration of a 1-form along a chain will be understood in the usual sense (see e.g. [**?**]), i.e. it is performed by taking the sum of the values of the form along that chain.

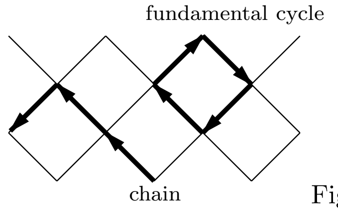

Figure 2

Likewise chains/cycles can be added in order to form new chains/cycles.

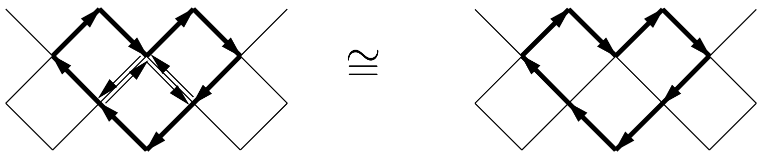

Figure 3

Definition 2.3 *Any cycle which is a finite sum of fundamental cycles is* **discrete homotopy equivalent** *to any fundamental cycle.*

A periodic light cone lattice $L_{2m,2n}$ may be viewed as $L/\mathbb{Z}$, where $\mathbb{Z}$ acts on the infinite light cone lattice L by shifts by $2m$ in space-like and $2n$, in time-like direction, where $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $(n < m)$ (cf. fig. 4). By the above notions a canonical Cauchypath $\mathcal{C}$ (cf. fig. 4) will be discrete homotopy **in**equivalent to a fundamental cycle.

Definition 2.4 *A one form with values in V whose integral is zero $\in V$ along any cycle being discrete homotopy equivalent to a fundamental cycle is* **closed.**

3 The phase space, symplectic current

The space of all solutions to an evolution difference equation with specified boundary conditions will be called covariant **phase space** $\mathcal{M}$. Let us choose periodic boundary conditions $g_k = g_{k+2p}$, the initial values $\left\{(g_k)_{k\in\{0,\dots 2p-1\}}\right\}$ along a Cauchypath $\mathcal{C}$ can then be viewed as coordinates on phase space $\mathcal{M}$ since any periodic solution $g \in \mathcal{M}$ is determined by its periodic initial conditions along the Cauchypath $\mathcal{C}$ (cf. fig. 4).

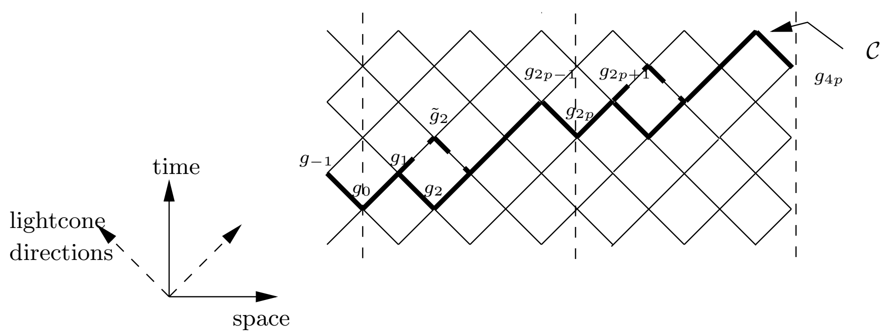

Figure 4

Denote with $\mathbf{d}$ the exterior derivative on $\mathcal{M}$. The following is a discrete version of an idea presented in [**?**].

Proposition 3.1 *The 1-form on $L_{2m,2n}$*

$$\omega : \mathcal{E} \to \Lambda^2(M^*) \tag{3.1}$$

$$(v_1, v_2) \to \mathbf{d}g_{v_1} \wedge \mathbf{d}g_{v_2} \tag{3.2}$$

$v_1, v_2 \in \mathcal{V}$ with values in the differential 2-forms on $\mathcal{M}$ is closed. ω is called a discrete symplectic current.

Proof:
By the evolution equation one has

$$\mathbf{d}\tilde{g}_k = V''(g_{k-1} - g_{k+1})(\mathbf{d}g_{k-1} - \mathbf{d}g_{k+1}) + \mathbf{d}g_k.$$

Hence

$$(\mathbf{d}\tilde{g}_k - \mathbf{d}g_k) \wedge (\mathbf{d}g_{k-1} - \mathbf{d}g_{k+1}) =$$
$$\mathbf{d}\tilde{g}_k \wedge \mathbf{d}g_{k-1} + \mathbf{d}g_{k-1} \wedge \mathbf{d}g_k + \mathbf{d}g_k \wedge \mathbf{d}g_{k+1} + \mathbf{d}g_{k+1} \wedge \mathbf{d}\tilde{g}_k = 0,$$

which is the integral of ω along a fundamental cycle. q.e.d.

Corollary 3.2 *The integral of ω along the Cauchypath $\mathcal{C}$ is a translational invariant presymplectic form on covariant phase space, which is compatible with evolution.*

Proof:
The integral of ω along the Cauchypath $\mathcal{C}$ is given by

$$\Omega = \Omega_j = \sum_{k=j}^{2p-1+j} \mathbf{d}g_k \wedge \mathbf{d}g_{k+1}.$$

Ω is clearly a closed 2-form on phase space $\mathcal{M}$. The integration is independent of the chosen domain, in other words Ω is translational invariant, i.e. $\Omega_j = \Omega_{j-1}$, $j \in \mathbb{Z}$. Ω is compatible with the evolution by construction. q.e.d.

Unfortunately Ω is degenerate, as having a two-dimensional null space. Nevertheless it is still possible to find a symplectic form, which is compatible with the evolution given in **??** - yet one has to enlarge the phase space $\mathcal{M}$.
A quasi-periodic field is a mapping $g : L \to \mathbb{R}$ with

$$g_{i+2p} - g_i = g_{i+2p+2} - g_{i+2} \quad \forall i,$$

i.e., there are two (space independent) monodromies $m_t^{(1)}, m_t^{(2)}$ defined by

$$m^{(i)} = g_{2p+2k+1-i} - g_{2k+1-i} \tag{3.3}$$

for an arbitrary $k \in \mathbb{Z}$. By the evolution equations **??** the monodromies are time independent. The phase space is now parametrized by the $2p+2$ variables $\{(g_k)_{k\in\{j,\ldots 2p+1+j\}}\}$.
Define the form

$$\Omega = \Omega_j = \sum_{k=j}^{2p-1+j} dg_{k+1} \wedge dg_k + \frac{1}{2}(dg_{j+2p+1} - dg_{j+1}) \wedge (dg_j + dg_{j+2p}) \quad (3.4)$$

Proposition 3.3 *Ω is a translational invariant symplectic form, which is compatible with the evolution.*

Proof:
The translational invariance can be proved by a direct check. The above form is invariant with repect to the (symplectic) map given by the elementary evolution at site k :

$$\tilde{g}_k = V'(g_{k-1} - g_{k+1}) + g_k \qquad \tilde{g}_l = g_l \quad l \in \{j \cdots 2p-1+j\} \quad l \neq k \quad l,k,j \in \mathbb{N}$$

which extends, by observing the time invariance of the monodromies, immediately beyond the fundamental domain (see fig. 4). The invariance under an such elementary evolution can be either checked directly or can be explained by the following. Choose a fundamental integration domain such that $j+1 < k < j+2p$ (this is always possible due to translational invariance). The boundary terms (terms with index $j, j+1, j+2p, j+2p+1$) in Ω are not affected by the elementary evolution and hence proposition **??** is applicable. Any evolution consists of a consecutive application of such elementary evolutions. q.e.d.

4 Lagrangian Description

For the case of a space-like periodic lightcone lattice $L_{2p,0}$ and $\mathcal{C}$ being a Cauchy zig-zag (as in fig. 1) the symplectic form in **??** was first derived in [**?**]. It was derived in [**?**] from a lagrangian action by using covariant phase space techniques [**?**],[**?**], [**?**]. The advantage of considering this special case lies in the fact that for $p = 2k$, $k \in \mathbb{N}$ it is compatible with a redefinition along the diagonals as described for equation (**??**). This isn't necessarily the case for general boundary conditions.

Let us briefly review the construction of a Lagrangian action as given in [**?**]. Define the following function $L : \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ on the fields evaluated along a Cauchy zig zag $\mathcal{C}$ (fig. 1):

$$L_j(v,w) = \sum_{k=j}^{p-1+j} v_k(w_k - w_{k+1}) + V(w_k - w_{k+1}) + w_j(v_{p-1+j} - v_{-1+j})(w_{p+j} - w_j).$$

v shall be a placeholder for fields, which are associated to the earlier (lower) row of vertices of $\mathcal{C}$ and w shall be a placeholder for fields, which are associated to the later (upper) row of vertices of $\mathcal{C}$. It is straightforward to check that on the set of quasiperiodic fields, i.e. fields for which:

$$v_{p+j} = v_j + m_v \qquad w_{p+j} = w_j + m_w$$

L_j is translational invariant, i.e. $L_j = L_{j+1} =: L$. Note that the translational invariance of L is less strong then the translational invariance of the corresponding symplectic structure in (**??**). L will be called a discrete Lagrangian for the dynamics as given in (**??**).

The Lagrangian action is to be obtained by summing up the Lagrangian L over all times. Hence the fields will be now labeled w.r.t. space and time as depictured in fig. 5.

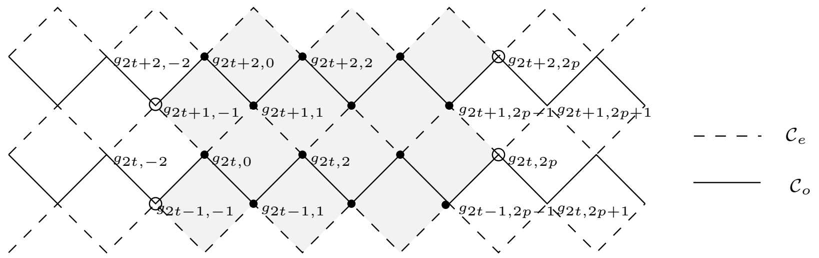

Figure 5

Via the identifications:

$$v_k = g_{2t,2k}, \qquad w_k = g_{2t+1,2k-1}$$

for the Cauchy path $\mathcal{C}_{\rceil}$ (fig. 5) and

$$v_k = g_{2t-1,2k+1}, \qquad w_k = g_{2t,2k}$$

for the Cauchy path $\mathcal{C}_{\wr}$ one time step earlier one obtains the following action:

$$\begin{aligned} S = \sum_t \Big\{ & \sum_{k=1}^{p-1} g_{2t,2k}(g_{2t+1,2k-1} - g_{2t+1,2k+1}) \\ & + g_{2t,0}(g_{2t+1,2p-1} - m_{2t+1}^{(2)} - g_{2t+1,1}) + g_{2t+1,2p-1} m_{2t}^{(1)} - \frac{1}{2} m_{2t}^{(1)} m_{2t+1}^{(2)} \\ & + \sum_{k=0}^{p-2} g_{2t-1,2k+1}(g_{2t,2k} - g_{2t,2k+2}) + g_{2t-1,2p-1}(g_{2t,2p-2} - g_{2t,0} - m_{2t}^{(1)}) \\ & + g_{2t,0} m_{2t-1}^{(2)} + \frac{1}{2} m_{2t}^{(1)} m_{2t-1}^{(2)} \\ & + \sum_{k=1}^{p-1} V(g_{2t+1,2k-1} - g_{2t+1,2k+1}) + \sum_{k=0}^{p-2} V(g_{2t,2k} - g_{2t,2k+2}) \\ & + V(g_{2t+1,2p-1} - m_{2t+1}^{(2)} - g_{2t+1,1}) + V(g_{2t,2p-2} - g_{2t,0} - m_{2t}^{(1)}) \Big\}. \end{aligned} \tag{4.1}$$

(**??**) is an extended version of the action given in [**?**]. As in [**?**] Variation with respect to $g_{t,k}, (t \in \mathbb{Z}, k \in 0, \dots 2p-1)$, yields the difference of the evolution equations (**??**) for the neighbouring faces to the left and to the right of $g_{t,k}$:

$g_{t,k}$ at even times:

$$\begin{array}{rcl} g_{2t+1,2k-1} - g_{2t-1,2k-1} & - & g_{2t+1,2k+1} + g_{2t-1,2k+1} \\ & - & V'(g_{2t,2k-2} - g_{2t,2k}) + V'(g_{2t,2k} - g_{2t,2k+2}) = 0, \end{array} \tag{4.2}$$

$g_{t,k}$ at odd times:

$$\begin{array}{rcl} g_{2t,2k} - g_{2t-2,2k} - g_{2t,2k+2} + g_{2t-2,2k+2} & - & V'(g_{2t-1,2k-1} - g_{2t-1,2k+1}) \\ & + & V'(g_{2t-1,2k+1} - g_{2t-1,2k+3}) = 0 \end{array} \tag{4.3}$$

The evolution equation (**??**) can be now obtained by considering the extended part of the action. Variation with respect to the monodromies $m_t^{(i)}$, $i = 1, 2$ yields:

$$\begin{array}{rcl} g_{2t+1,2p-1} - g_{2t-1,2p-1} - V'(g_{2t,2p-2} - g_{2t,0} - m_{2t}^{(1)}) & = & \frac{1}{2}m_{2t+1}^{(2)} - \frac{1}{2}m_{2t-1}^{(2)} \\ g_{2t+2,0} - g_{2t,0} - V'(g_{2t+1,2p-1} - m_{2t+1}^{(2)} - g_{2t+1,1}) & = & \frac{1}{2}m_{2t}^{(1)} - \frac{1}{2}m_{2t+2}^{(1)}. \end{array} \tag{4.4}$$

Now, as we may write the monodromy as the sum of differences of field variables:

$$m_t^{(i)} = \sum_{k=1}^{p} (g_{t,2k+1-i} - g_{t,2k-1-i})$$

equations (**??**) are sufficient to enforce the time independence of the monodromies. Hence, the right hand sides of equations (**??**) vanish. We get the evolution equations (**??**) for the faces "above" $g_{2t-1,2p-1}$ and $g_{2t,0}$ for all t, and thus finally for all faces.

The above difference variables for fixed monodromies parametrize the Marsden-Weinstein reduced phase space, for details see [**?**].

Bibliography

[A74] V. I. Arnold, "Mathematical Methods of Classical Mechanics", snd ed. Springer Verlag (1989)

[BHS91] G. Barnich, M. Henneaux, C. Schomblond, "Covariant description of the canonical formalism", *Phys. Rev. D* **44**, R939-R941 (1991)

[BBR96] V. Bazhanov, A. Bobenko, N. Reshetikhin, "Quantum Discrete Sine-Gordon Model at Roots of 1: Integrable Quantum System on the Integrable Classical Background", *Comm. Math.Phys.* **175** 377–400 (1996)

[BP96] A.I. Bobenko, U. Pinkall, "Discrete Surfaces with Constant Negative Gaussian Curvature and the Hirota Equation", *J. Diff. Geom.***43** No. 3, 527–611 (1996)

[BP97] A.I. Bobenko, U. Pinkall, “Discretization of surfaces and integrable systems”, this volume

[BKP93] A.I. Bobenko, N. Kutz, U. Pinkall, “The Quantum Pendulum”, *Phys. Lett. A* **177**, 399 – 404 (1993)

[BRST91] M. Bruschi, O. Ragnisco, P.M. Santini, G.-Z. Tu, “Integrable Symplectic Maps”, *Physica D* **49**, 273 (1991)

[CW87] C. Crnković, E. Witten, “Covariant Description of canonical formalism in geometrical theories”, in “Three hundred years of gravitation” *Cambridge University Press* Ed.: Hawking, Israel, 676–684 (1987)

[EK95] C. Emmrich, N. Kutz, “Doubly discrete Lagrangian systems related to the Hirota and sine-Gordon equation”, *Phys. Lett. A* **201** 156–160 (1995)

[FV92] L.D. Faddeev, A.Yu. Volkov, “Quantum Inverse Scattering Method on a Spacetime Lattice”, *Theor. Math. Phys.* **92**, 837–842 (1992)

[FV94] L.D. Faddeev, A.Yu. Volkov, “Hirota Equation as an Example of an Integrable Symplectic Map”, *Lett. Math. Phys.* **32** 125–136 (1994)

[Hi77-3] R. Hirota, “Nonlinear Partial Difference Equations III; Discrete Sine-Gordon Equation”, *J. Phys. Soc. Japan* **43**, 2079-2086 (1977)

[K94] N. Kutz, “On the spectrum of the quantum pendulum”, *Physics Letters A* **187**, 365-372 (1994)

[KKS97] J. Kellendonk, N. Kutz, R. Seiler, “Spectra of Quantum Integrals”, this volume

[NCP92] F.W. Nijhoff, H.W. Capel, V.G. Papageorgiu, “Integrable Quantum Mappings”, *Phys. Rev. A* **46**, 2155 (1992)

[CN94] H.W. Capel, F.W. Nijhoff, “Integrable Quantum Mappings ” in: *Symmetries and Integrability of Difference Equations*, Eds. D. Levi, L. Vinet and P. Winternitz, CRM Proceedings and Lecture Notes, vol. **9l**, (AMS, Providence 1996), pp. 37–49 (1996)

[S89] Yu. B. Suris, “Integrable Mappings of the Standard Type”, *Funct. Anal. Appl.* **23**, 74-79 (1989)

[V91] A.P. Veselov, “Integrable Maps”, *Russ. Math. Surv.* **46** No. **5** (1991)

[Z86] G. Zuckerman, “Action Principles and Global Geometry”, *Math. Aspects of String Theory* Ed.: Yau (1986)